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ARTICLES

MEETINGS

MISCELLANEOUS

ANNUAL SURVEY OF ECONOMIC THEORY:
THE THEORY OF MONOPOLY

By J. R. HICKS

I PROPOSE in this survey to confine attention to the progress which has
recently been made in one particular part of economic theory. Such a
limitation has obvious advantages in facilitating more detailed dis-
cussion; and when one has decided to confine oneself to a particular
field, it is obvious that monopoly has the best claim to be chosen. The
last five or six years have seen the appearance of at least four im-
portant works specially devoted to this subject—those of Dr. Zeuthen,
Dr. Schneider, Professor Chamberlin, and Mrs. Robinson;[1] while
there is, I think, no theoretical subject which has received more at-
tention in the recent volumes of most of the chief economic journals
than the theory of monopoly and imperfect competition. To most of
these articles we shall refer as we proceed; but the names of Mr. Har-
rod, Mr. Shove, Dr. v. Stackelberg, and Professor Hotelling, cannot be
omitted from even a preliminary bibliography.[2]

The preoccupation of contemporary theorists with problems of
monopoly does not appear to be due, as might perhaps be expected,
to their consciousness of the increased urgency of these problems in the
modern world. It may very well be that monopoly is more important
today than it was fifty years ago, though it is not so obvious as it ap-
pears at first sight. It is certain, however, that the phenomena of
monopolistic competition to which attention has so particularly been
directed are not new phenomena; they were observed and analyzed,
however imperfectly, by older economists, by Cairnes and Wicksell,
if by no others.[3]

The widespread interest in monopoly theory is much easier to ac-
count for on grounds inherent in the development of economic theory
itself, though here an element of coincidence is present. On the one
hand, the generally increased interest in mathematical economics dur-

[1] F. Zeuthen, *Problems of Monopoly and Economic Warfare*, London 1930;
Schneider, *Reine Theorie monopolistischer Wirtschaftsformen*, Tübingen 1932;
E. H. Chamberlin, *Theory of Monopolistic Competition*, Harvard 1933; J. Robin-
son, *Economics of Imperfect Competition*, London 1933.

[2] R. F. Harrod, "Notes on Supply," *Econ. Jour.* 1930; "Law of Decreasing
Costs," *E.J.* 1931; "Doctrines of Imperfect competition" *Q.J.E.* 1934; G. F.
Shove, "The Imperfection of the Market," *E.J.* 1933; H. Hotelling, "Stability
in Competition," *E.J.* 1929.

[3] Cairnes, *Political Economy*, pp. 115–116 (quoted Chamberlin, *op. cit.* p.
106); Wicksell, *Lectures on Political Economy*, I, pp. 87–88.

ing the last few years (of which this journal is itself a symptom) has naturally turned attention back to the work of Cournot, the great founder of the subject, and still one of its best teachers. It was Cournot's creation of elementary monopoly theory which was the first great triumph of mathematical economics; yet Cournot had left much undone, and it is not surprising that the endeavor to complete his work should have been an attractive occupation for his successors.

But if some modern monopoly theorists have been seeking to fill the gaps in Cournot, others have been more concerned with the gaps in the work of Marshall. These gaps were more skilfully passed over, and it was not until after many years' criticism that they were clearly discerned. But the controversy on the "Laws of Returns," begun by Mr. Sraffa in 1926, and carried on more or less continuously in the *Economic Journal* for some years afterwards,[4] made it increasingly evident to the most convinced Marshallians that the device of "external economies," by which Marshall sought to reconcile the postulate of perfect competition with the observed facts of increasing returns, would not bear the weight that had been imposed upon it. A tendency therefore developed away from the postulate of perfect competition. The participants in the discussion began to assume as the normal case that a firm can influence to some extent the prices at which it sells, that it is confronted with a downward sloping demand curve for its products, though this demand curve may have a high elasticity. With this assumption, the cardinal difficulty of increasing returns disappeared, since a firm might still be in equilibrium under conditions of diminishing cost. But numerous other difficulties started up, and it became necessary for these writers, like those mentioned before, to make a detailed examination of the theory of monopoly.

From each line of approach a substantially similar theory has emerged, though there are important points which still remain controversial. It remains convenient for us to discuss the modern theory under the old headings: (1) *Simple monopoly*, where the individual firm is confronted with given demand functions for its products, and given supply functions for its factors; (2) *Monopolistic Competition*, the relations of a group of firms producing similar products, i.e., an industry; (3) *Bilateral Monopoly*, where one firm is selling to another.

I. SIMPLE MONOPOLY

As far as simple monopoly is concerned, the improvement on Cournot is mainly a matter of exposition, although there has been some further enquiry into the effect of monopoly on the demand for factors of production.

[4] See bibliography in *Economic Journal* (1930), p. 79.

1. If the prices at which the monopolist hires his factors are fixed, his cost of production can be taken as a simple function of output. Let $\phi(x)$ be the total cost of producing an output x.

If the monopolist's selling price is p, and $p = f(x)$ is the demand curve confronting him, his profit on selling an output x will be

$$xf(x) - \phi(x)$$

which is maximised when

$$xf'(x) + f(x) = \phi'(x).$$

So much has been familiar since Cournot; the principal recent innovation has been to give the expression on the left of the last equation a name "Marginal Revenue."[5] The equation can then be written

Marginal Revenue = Marginal Cost

which is certainly a convenient way of expressing the first condition of monopolistic equilibrium.

Since the elasticity of the demand curve $= \eta = -\dfrac{f(x)}{xf'(x)}$, marginal revenue

$$= \text{price} \left(1 - \frac{1}{\eta} \right).$$

The second condition of maximum profits is that

$$\frac{d}{dx} \big\{ xf'(x) + f(x) - \phi'(x) \big\}$$

should be negative. This can be written

$$\frac{d}{dx} (MR) < \frac{d}{dx} (MC).$$

Monopolistic equilibrium is therefore stable, so long as the marginal revenue curve slopes downwards more steeply than the marginal cost curve. All cases where the marginal revenue curve slopes downwards and the marginal cost curve upwards are therefore stable, but instability may occur if either of these conditions is not fulfilled. Upward sloping marginal revenue curves, though possible, are unlikely to be very important, since the demand curve from which a marginal revenue

[5] So Mrs. Robinson. It seems the most convenient of the names which have been suggested.

curve is derived may be taken to be always downward sloping. Much more important is the fact that stable equilibrium with a downward sloping marginal cost curve is possible, so long as the downward slope is less than that of the marginal revenue curve, and so long, also, as total receipts exceed total costs by an amount sufficient to keep the monopolist in business.

The question of stability once settled, it becomes possible to apply the apparatus in the ordinary manner, familiar in elementary theory, to simple problems of change. A rise in the marginal cost curve will reduce output, a rise in the marginal revenue curve will increase it; but a rise in the demand (average revenue) curve may not increase output, unless it is such as to cause a rise in the marginal revenue curve. Similarly a rise in average costs will not contract output, unless it is associated with a rise in marginal costs, or is otherwise large enough to drive the monopolist out of business.

2. *The monopolist and the factors of production*.[6]—It is convenient, for the analysis of this problem, to conceive of the monopolist as owning certain factors of production (his *private factors*, we may call them) and hiring others. If he is unable to vary the supply of these private factors, then it is strictly correct to suppose him endeavouring to maximize his profits, that is to say, to maximise the net earnings of these private factors. If this assumption cannot be made, difficulties emerge, which had better be examined later.

If the quantities of factors hired are a, b, c, $\cdots$, their prices π_a, π_b, π_c, $\cdots$, and their supply curves to the monopolist are given, then

$$\text{Monopoly profit} = xp - a\pi_a - b\pi_b - c\pi_c - \cdots.$$

This is maximised when

$$\left(p + x\frac{dp}{dx}\right)dx - \left(\pi_a + a\frac{d\pi_a}{da}\right)da - \left(\pi_b + b\frac{d\pi_b}{db}\right)db - \cdots = 0$$

which becomes

$$MR\,dx - MC_a\,da - MC_b\,db - \cdots = 0,$$

if we write MC_a for $\pi_a + a\dfrac{d\pi_a}{da}$, and so on.

Taking $x = \phi(a, b, c, \cdots)$ as the production function, technically given, then

[6] Robinson, *op. cit.*, Books VII–IX; Schneider, "Bemerkungen zur Grenzproductivitätstheorie," *Zeitschrift für Nationalökonomie*, 1933. See also Dr. Schneider's *Theorie der Produktion* (1934); pp. 57, 76.

$$dx = \frac{\partial x}{\partial a}\, da + \frac{\partial x}{\partial b}\, db + \cdots.$$

Substituting in the above, we have

$$\left(MR\, \frac{\partial x}{\partial a} - MC_a \right) da + \left(MR\, \frac{\partial x}{\partial b} - MC_b \right) db + \cdots = 0.$$

Since this equation must hold for all values of da, db, $\cdots$, it follows that

$$MR\, \frac{\partial x}{\partial a} = MC_a, \quad MR\, \frac{\partial x}{\partial b} = MC_b, \cdots$$

for all factors.

MC_a, MC_b, $\cdots$, are the *marginal costs to the monopolist* of hiring an additional unit of the factors, a, b, $\cdots$. If the supply curves of the factors slope upwards, these marginal costs will exceed the prices of the factors by $a\,\dfrac{d\pi_a}{da}$ etc., respectively, that is to say, by the additional amounts which have to be paid on earlier units in order to keep their prices on a level with that of the marginal unit of the factor. $MR\,\dfrac{\partial x}{\partial a}$ is conveniently described as the "marginal value product" of the factor a; it is the increment in the total value of the product which results from the application of an additional unit of a. The condition of factor equilibrium is thus that the marginal value product of a factor should equal its marginal cost.

The stability conditions for factor equilibrium do not appear to have been fully investigated; but a cursory examination suggests that there are several ways in which the presence of monopoly brings into the possible range of stable equilibria positions which would not be stable under perfect competition.

If the supply curve of any factor to the monopolist is horizontal, so that the monopolist is unable to affect the price of that factor, then even so his demand for that factor will be reduced below what it might have been, if the product demand curve confronting him is imperfectly elastic. Monopolistic exploitation of the consumer therefore brings about a directly consequent reduction in the demand for factors. And if a number of monopolists are employing a particular factor, they may each be unable by isolated action to influence the price of the factor; and yet, in their efforts to exploit the consumer, they will

each reduce their demand for the factor, and the price of the factor may, in consequence, be reduced. But this is a different thing from the additional reduction in demand which comes about if a monopolist is able to influence the price of a factor directly, so that he takes into account the saving on other units which he gets by reducing his demand at the margin. The first type of reduction would be called by Mrs. Robinson "monopolistic exploitation" of the factor, while she has invented the term "monopsonistic" to describe exploitation of the second type.

3. *Simple monopoly and joint production.*—Nearly all the writers here discussed have confined their analysis of simple monopoly to the case where the monopolist produces only one product.[7] For reasons which will appear later, this limitation seems rather unfortunate. A brief but illuminating discussion of the problem has, however, been given by Dr. von Stackelberg, which we may here reproduce.[8]

It is convenient, in order to isolate the problem, to assume that the prices of the factors are now given to the monopolist; we can then introduce a cost function expressing the total cost of production of quantities $x_1, x_2, \cdots$, of the different products. Let $\phi(x_1, x_2, \cdots)$ be the cost function.

Then Monopoly profit $= p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots - \phi(x_1, x_2, \cdots)$.

If we assume that the demand curves for the various products are independent, so that p_1 depends upon x_1 only, not on $x_2, x_3 \cdots$, then the conditions of equilibrium are

$$\frac{d}{dx_1}(p_1 x_1) = \frac{\partial \phi}{\partial x_1}, \quad \frac{d}{dx_2}(p_2 x_2) = \frac{\partial \phi}{\partial x_2}, \cdots.$$

The ordinary "marginal revenue marginal cost" condition still holds.

If, however, the demand curves are not independent, then the conditions become

$$p_1 + x_1 \frac{\partial p_1}{\partial x_1} + x_2 \frac{\partial p_2}{\partial x_1} + \cdots = \frac{\partial \phi}{\partial x_1}$$

$$p_2 + x_1 \frac{\partial p_1}{\partial x_2} + x_2 \frac{\partial p_2}{\partial x_2} + \cdots = \frac{\partial \phi}{\partial x_2}$$

and so on. That is to say, the monopolist has to take into account, when fixing the output of any particular product, not only the reaction

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[7] Professor Chamberlin gives us an interesting account of the factors which determine what that one product shall be (*op. cit.* ch. 4 and 5).

[8] H. von Stackelberg, *Grundlagen einer reinen Kostentheorie* (Vienna 1932), p. 68. See also Hotelling, "Edgeworth's Taxation Paradox," *Journal of Political Economy* 1932.

of an increased supply upon the price of that product, but also its re-action upon the prices of all other products which he is selling. If the cross-coefficients $\left(\dfrac{\partial p_2}{\partial x_1} \text{ etc.}\right)$ are negative (roughly speaking, the case when the different products are competitive in consumption),[9] these reactions will lower the marginal revenue curve for any particular product, and so tend to restrict output. But in the opposite case, when the cross-coefficients are positive, the marginal revenue curve will be raised; so that here the restriction of output under monopoly will be less than we should have at first expected.

If $x_2 \dfrac{\partial p_2}{\partial x_1} + x_3 \dfrac{\partial p_3}{\partial x_1} + \cdots$ is positive, and greater than $\dfrac{\partial \phi}{\partial x_1}$, it may

pay the monopolist to produce a finite output of x_1, even if he has to give it away. And such a phenomenon is surely not uncommon; a very considerable part of what are usually described as "selling costs" comes very conveniently under this head. The subject of selling costs has been analyzed at considerable length and with much insight by Professor Chamberlin, who maintains, however, the single-product firm as the foundation of his analysis. It may be suggested that the subject could be further illuminated, and brought closer into relation with fundamentally analogous cases where the "bait" is not actually given away, if a start had been made from Dr. von Stackelberg's more general case.[10]

4. *Discrimination.*—From one point of view, discrimination is a limiting case of joint production. When we say that a single commodity is sold by a monopolist at various different prices, the singleness of the commodity consists solely in its various units being perfect substitutes on the supply side. We can introduce this condition of being perfect substitutes in production, and so go over from joint production to discrimination.

But this line of approach, although it has conveniences, and brings discrimination into a very satisfactory relation with general monopoly theory, is not that which has traditionally been adopted. Of recent writers, Mrs. Robinson is the only one who has added anything substantial to the traditional theory of discrimination. She has devoted to it what is probably the best, as it is certainly the most ingenious, part

[9] I say "roughly speaking," for it is becoming apparent that the terms *competitive* and *complementary* conceal a great many ambiguities. (See Hicks and Allen, "A Reconsideration of the Theory of Value," *Economica* 1934.)

[10] The same foundation might be used for an analysis of monopolistic exploitation by "compulsory joint supply."

of her book; there can be no question that these chapters will find their place along with Dupuit and Pigou on the very select bibliography of discrimination theory.

5. *The "private" factors.*—Most modern writing on monopoly, as we have said, has been content to assume a monopolist simply seeking to maximise his profits, that is to say, it neglects possible changes in the supply of private factors. This omission seems to me unfortunate, though it must be confessed that the subject presents grave difficulties.[11] On the one hand, unless we assume that the marginal utility of money to the monopolist is constant, we cannot unambiguously express in monetary terms the subjective cost to the monopolist of producing additional units of output; we are therefore unable to introduce the private factors into the "marginal revenue = marginal cost" equation, and are obliged to fall back upon Paretian indifference curves, more cumbrous, and in this case decidedly less informative. The second difficulty is even more formidable. Under conditions of monopoly, there is no reason to suppose any particular connection between subjective cost and *output*, since it is probable that a considerable part of the monopolist's efforts and sacrifices will be devoted, not to increasing his output, but finding to what precise point he should restrict it. Now, as Professor Bowley[12] and others have pointed out, the variation in monopoly profit for some way on either side of the highest profit output may often be small (in the general case, it will depend on the difference between the slopes of the marginal revenue and marginal cost curves); and if this is so, the subjective costs involved in securing a close adaptation to the most profitable output may well outweigh the meagre gains offered. It seems not at all unlikely that people in monopolistic positions will very often be people with sharply rising subjective costs; if this is so, they are likely to exploit their advantage much more by not bothering to get very near the position of maximum profit, than by straining themselves to get very close to it. The best of all monopoly profits is a quiet life.

II. MONOPOLISTIC COMPETITION

1. We come now to the "group problem," the equilibrium of a group of firms producing similar but not identical products. The treatment of this problem by Professor Chamberlin and by Mrs. Robinson (the same applies, though with some qualification, to Mr. Harrod) is based upon a very neat geometrical proposition.[13] Since the products of the

[11] Cf. Robinson, "Euler's Theorem and the Problem of Distribution," (*E. J.*, 1934).

[12] *Mathematical Groundwork of Economics*, pp. 25, 60.

[13] Chamberlin, *op. cit.* p. 84; Robinson, pp. 94–95; Harrod, "Doctrines of Imperfect Competition," *Q.J.E.* 1934, p. 457.

various firms are not identical, the demand curve which confronts each individual firm will not be horizontal, but will slope downwards.[14] On the other hand, if entry into the industry is free, it will be impossible for the firms in the industry to earn more than "normal profits." On the basis of the first assumption, it is concluded that the output of each firm will have to satisfy the condition of monopolistic equilibrium, marginal revenue = marginal cost. On the basis of the second, it is concluded that the price of each product will have to equal average cost, when average cost is calculated in such a way as to include "normal profits."

If then we write π_x = average cost (in the above sense) of producing an output x, and p_x = the price at which the firm can sell that output, the second condition gives us

$$p_x = \pi_x \tag{1}$$

while we have from the first condition

$$\frac{d}{dx}(xp_x) = \frac{d}{dx}(x\pi_x)$$

$$\therefore \quad p_x + x\frac{dp_x}{dx} = \pi_x + x\frac{d\pi_x}{dx}$$

$$\therefore \text{from (1)}, \frac{dp_x}{dx} = \frac{d\pi_x}{dx}. \tag{2}$$

From (1) and (2) it follows that the demand curve and the average cost curve must touch at a point of equilibrium.

Since the demand curve is downward sloping, the average cost curve must also be downward sloping at the equilibrium point. Equilibrium under monopolistic competition is only possible when average costs are diminishing; that is to say, the equilibrium output of a firm will be less than the output which would give minimum average costs—the output which would actually be reached under conditions of perfect competition. From this Professor Chamberlin proceeds to the conclusion that analysis based on perfect competition makes "the price always too low, the cost of production too low, the scale of production too large, and the number of producers too small."

In order for us to estimate the importance of this result, we must begin by examining the premises on which it is based. To take first the "average cost curve." When Walras and Pareto reckoned profits into

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[14] Professor Chamberlin constructs this individual demand curve on the assumption that the prices of the rival commodities remain unchanged (p. 75). Mrs. Robinson's formulation seems distinctly ambiguous (p. 21).

costs, they were thinking of conditions of perfect competition, and
their conclusion that price = average cost, so that the entrepreneur
makes "ni bénéfice ni perte," meant nothing else than that the private
factors of the entrepreneur could get no other return in the static
equilibrium of perfect competition than would have accrued to them
if they had been directly hired out on the market. But is it possible to
transfer this conception to the theory of monopolistic competition?
So far as the private factors are to some extent unique, so that there
are no perfect substitutes for them (and this seems the most likely
case in which monopolistic competition might arise), they can have no
market price which is not to some extent monopolistically determined.
If there are perfect substitutes for them, why are those perfect sub-
stitutes not being employed in making perfect substitutes for the
product?

There is only one way out of this dilemma, and I can only suppose
that it is this which the writers in question have in mind. The factors
of production, private or hired, may be sufficiently divisible, and suffi-
ciently scattered in ownership, to ensure that there is a perfect market
for them, or something sufficiently perfect for the imperfections to be
negligible. But there may still be a range of increasing returns in the
production of any particular product, due to indivisibilities in the pro-
duction function, not in the factors themselves.[15] If this is the case,
substantially homogeneous factors may be put together by a limited
number of firms into a limited number of different products, each of
which is unique, and the demand curve for each of which is downward
sloping.

This is the only state of affairs of which the Chamberlin-Robinson
apparatus seems to be an exact description; it is probable that it does
correspond with a certain region of reality. But I cannot help feeling
that the application of the apparatus is implicitly much exaggerated.
This is only partly because of the actual heterogeneity of factors—
both writers accept this difficulty, and at the worst it only means that
the technical apparatus is over-rigid. They can still claim to have
shown that monopolistic restriction of output is compatible with earn-
ings in no way out of the ordinary. A much more serious objection
arises from the variability of the product.

There are two relevant sorts of product variation. One, the only
kind which has been much discussed, is where each firm produces a
single product, but the nature of that product is capable of being
changed. This problem has been dealt with mostly in terms of location;

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[15] Kaldor, "The Equilibrium of the Firm," *Econ. Jour.* March 1934, p. 65n.
On the general question of indivisibilities and costs, see also the appendix to
Mrs. Robinson's book; also Schneider, *Theorie der Production*, ch. 1.

a product available in a different place is economically a different product, and a change in the location of the firm is one of the ways of varying the product. (Professor Chamberlin's discussion of location is, however, reinforced by a discussion of the same problem in more general terms.)

In his paper, "Stability in Competition,"[16] Professor Hotelling had demonstrated that there is a tendency, when two firms are competing for a given market, for them to get together in the centre of the market. This tendency in itself would thus be favorable to the establishment of conditions of approximately perfect competition, if it could be shown to hold for more firms than two.

Unfortunately, as Professor Chamberlin shows, this is not so.[17] Once there are more than two firms in the market, they will tend to scatter, since any firm will try to avoid being caught between a pair of others. It seems evident that this general tendency to dispersion will be present when it is a question of quality competition as well as of competition in location, though of course the possible kinds of variation are even more complex.

Thus, so long as we retain the "one firm one product" assumption, variability of the product is not sufficient to prevent an appreciable degree of imperfection in the elasticity of the demand curve confronting any particular firm. The position seems, however, to be different once we drop this assumption.

In fact, when "product" is interpreted in the strict economic sense of a collection of articles that are to the consumer perfect substitutes, almost every firm does produce a considerable range of different products. It does so largely because there are economies to be got from producing them together,[18] and these economies consist largely in the fact that the different products require much the same overheads. Further, at any time the products it is actually producing will probably not exhaust the list of products it could produce from approximately the same plant. Thus it will have various potential products which it could produce in small quantities at quite a low marginal cost.

Now when other producers are able to supply small quantities of highly competitive products at low prices, this is at last an effective force tending to keep the demand curve for a particular product of a particular firm very highly elastic. Of course, it will probably not be

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[16] *Op. cit.* See also Zeuthen, "Theoretical Remarks on Price Policy," *Q.J.E.* 1933.

[17] *Theory of Monopolistic Competition*, Appendix C.

[18] In the sense that it costs less to produce outputs x_1 and x_2 in a single firm, than it would cost (in total) to produce output x_1 in one firm and output x_2 in another.

perfectly elastic; for in fact any degree of specialization on a particular line offers a *prima facie* case that the specializing firm has some particular facilities for that line, and it may be able to carry out a certain a degree of restriction before it tempts other firms to follow it. Further, a firm is always likely to be on the lookout for a line in which it is relatively safe from such competition. Nevertheless, this consideration does seem to go a good way to justify the traditional practice of economists in treating the assumption of perfect competition as a satisfactory approximation over a very wide field.[19]

A considerable degree of the sort of market imperfection we have been discussing seems likely to arise in two cases only: (1) where the producer has command of some specialized "factor," such as patent, legal privilege, site, or business capacity, for which no clear substitute is available; (2) where economies of scale are narrowly specialized, so that it would be impossible for another firm to produce commodities highly competitive with these produced by the first firm excepting at much greater marginal cost. There is no doubt that such conditions as these are fairly frequent, but they are, after all, precisely the cases which have been traditionally treated under the heading of monopoly.

2. *Duopoly.*—There is, however, one further difficulty of great importance. We have suggested that the demand curve for a particular product of a particular firm will usually be kept highly elastic by the incursion of other producers selling small quantities of highly competitive products, if the first firm raises its price. But if they do so, will not the first firm retaliate on them?

Two cases have thus to be distinguished. The first is when the other potential producers are fairly numerous. In this case, they are not likely to be much deterred by the fear of retaliation. For although the first firm may find it profitable to turn its attention to some other product if it meets with competition in the line it had first chosen, the chance of that other product being highly competitive with the products of any particular other producer is small.

In the other case, when the other potential producers are few, the fear of retaliation is likely to be more serious, and it may very well stop poaching.

The difficult problem which arises from the relations of a very small number of competing firms has been much studied in recent years, but there has not yet developed any very close agreement on the solution. Largely owing to the difficulty of the problem, it has been chiefly studied in its most simple case, that of two firms producing an identical product—duopoly.[20]

[19] Cf. Shove, "The Imperfection of the Market," *Econ. Jour.* 1933, pp. 115–116.

The theory of duopoly has a long history; and here we can do no more than allude to the classical theory of Cournot, and the displacement of Cournot's theory by the criticisms of Bertrand and Edgeworth, which form the ancient history of the subject. Edgeworth's solution, based on "the characteristic freedom of the monopolist to vary price," involved such peculiar assumptions about costs that it could hardly have held the field forever. The post-war period therefore saw a renaissance of Cournotism, led by Amoroso and Wicksell;[21] this movement is represented also by the chapter on "Mehrfaches Monopol" in Dr. Schneider's book.[22] In the next stage, criticisms of both the Cournot and Edgeworth solutions were offered by Dr. Zeuthen and by Professor Chamberlin;[23] it then became clear that each of the rivals had pointed the way towards a possible solution, but that even together they did not exhaust the list.

A very convenient line of approach, which sets these alternative solutions in their places, and so opens a path towards a general theory, can be developed from a hint given in Professor Bowley's *Mathematical Groundwork*.[24] It is this approach which appears to be gaining ground at present. Its main principle can be expressed as follows.[25]

The marginal revenue, which a duopolist endeavors to equate to his marginal cost,

$$= \frac{d}{dx_1} (px_1)$$

where x_1 is his output, and $p = f(x_1 + x_2)$, x_2 being the output of his rival. Thus

$$MR_1 = \frac{d}{dx_1} (px_1) = p + x_1 f'(x_1 + x_2) + x_1 f'(x_1 + x_2) \frac{\delta x_2}{\delta x_1}.$$

The marginal revenue curve which confronts the duopolist is thus in

part dependent upon a quantity $\dfrac{\delta x_2}{\delta x_1}$, which we can only interpret

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[20] Chamberlin, however, has made at any rate a preliminary investigation of the more complex cases where several firms are involved. See his sections on "oligopoly" (*Theory*, pp. 100, 170).

[21] Amoroso, *Lezioni d'economia matematica;* Wicksell, Review of Bowley's *Mathematical Groundwork, Archiv für Sozialwissenschaft* 1927.

[22] Schneider, *Reine Theorie*, ch. 4.

[23] Zeuthen, *Problems of Monopoly*, ch. 2; Chamberlin, *Theory*, ch. 3, which substantially reproduces his article on "Duopoly," *Q.J.E.* 1929.

[24] P. 38.

[25] The following owes much to some yet unpublished work by Mr. W. M. Allen, of Oxford.

as the degree to which the duopolist expects his rival to expand (or contract) output, if he himself expands his output by an increment dx_1. Since $f'(x_1+x_2)$ is negative, a negative value of $\dfrac{\delta x_2}{\delta x_1}$ will raise

the adjusted marginal revenue curve of the duopolist, and thus be favorable to an expansion of output; a positive value will favor a contraction.

The conception of these "conjectural variations," $\dfrac{\delta x_2}{\delta x_1}$ etc., has been

analysed in very general terms by Professor Frisch.[26] There is, in the short period, no need for any particular degree of consistency between

the conjecture of the first duopolist $\dfrac{\delta x_2}{\delta x_1}$, and that of the second $\dfrac{\delta x_1}{\delta x_2}$.

The equation of marginal revenue and marginal cost thus determines the output of the first duopolist, once the output of the second duopolist, and the first duopolist's conjecture as to the variation of this output are given. For any particular type of conjecture, we can thus construct a "reaction curve," similar to that employed by Cournot, giving the preferred output of the first duopolist, corresponding to each possible output of the second. A similar reaction curve can be constructed for the second duopolist, and the intersection of the two will give the point of equilibrium.

In the majority of cases, these reaction curves will be negatively inclined;[27] and in the majority of these cases, the inclination will be such that an increased output by the other duopolist will react on the first in such a way as to increase the total output of both together. If we

[26] "Monopole—Polypole—La Notion de Force dans l'économie," *Nationaløkonomisk Tidsskrift* 1933.

[27] The condition for negative inclination is that $1 + \dfrac{hx_1}{x}\left(1+\dfrac{\delta x_2}{\delta x_1}\right)$ should be positive; where h is the "adjusted concavity" of the market demand curve. (That is to say, $h = \dfrac{(x_1+x_2)f''(x_1+x_2)}{f'(x_1+x_2)}$. Cf. Robinson, *Economics of Imperfect Competition*, p. 40.) Since we may assume that in all sensible cases, $1 + \dfrac{\delta x_2}{\delta x_1}$ is positive, it follows that the reaction curve will be negatively inclined in all cases when h is positive (when the demand curve is convex upwards) and also for a considerable number of cases when h is negative. It has been further shown by Mr. Allen that in such cases of negative inclination, the slope of the reaction

confine our attention to these *normal* cases, which are much the most likely to yield stable solutions, the more interesting assumptions about conjectures which have been made by recent writers fall into their places very simply.

(1) If the conjectural variations are both zero, we have of course the Cournot case. (2) If one of the conjectural variations is zero, but the other duopolist takes as his conjectural variation the actual slope of the reaction curve of his rival, we have the case of an "active" policy by one duopolist.[28] In *normal* conditions, this will make the conjectural variation of the active duopolist negative; thus, as compared with the Cournot case, it will raise his marginal revenue curve, increase his output, and (again in normal conditions) lead to an increased total output, and so a lower price. (3) If both duopolists act in this manner, each calculating conjectural variations from the other's Cournotian reaction curve, we have a curious case which has been investigated by Dr. von Stackelberg and Mr. Harrod.[29] In normal conditions, once more, this will lead to a further expansion of total output, and a further fall in price. (4) There does not seem to be any reason why we should stop here. One duopolist may become doubly "active," and calculate a conjectural variation from the reaction curve of his rival on the assumption that the rival is active. In most, though not (it appears) quite all, *normal* cases, this would lead to a further fall in price. The process becomes similar to one of price-cutting.

But once we are on the road of competitive price-cutting, it is reasonable to suppose that, sooner or later, one duopolist or the other would perceive that his conjecture that an increase in his output was leading to a contraction of his rival's was proving wrong. Once he acted on this, and constructed a conjectural variation based on this experience (and consequently a *positive* variation) the whole situation would be transformed. Price-cutting would give place to "tacit combination"; positive conjectures, again in normal conditions, would give a higher price than that given by the Cournot equilibrium.[30]

The method just described is capable of extension to the case where

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curve will also (for reasons of stability) be numerically less than 1, excepting when there is a high degree of asymmetry between the positions of the two duopolists. "Normal cases" are defined as satisfying these two conditions, so that dx_1/dx_2, taken along the reaction curve of the first duopolist, lies between 0 and -1.

[28] v. Stackelberg, "Sulla teoria del duopolio e del polipolio," *Rivista italiana di statistica*, June 1933. This article also contains an important and ingenious extension of the theory to the case of several producers.

[29] v. Stackelberg, *ibid*. Harrod, "The Equilibrium of Duopoly," *Economic Journal*, June 1934.

[30] Nicoll, "Professor Chamberlin's Theory of Limited Competition," *Q.J.E.*

the product of one duopolist is not a perfect substitute for that of the other. We have only to write $p_1 = f_1(x_1, x_2)$, $p_2 = f_2(x_1, x_2)$; the two sellers will now of course usually sell at different prices. We then have

$$\text{Adjusted marginal revenue of first seller} = \frac{d}{dx_1}(p_1 x_1) = p_1 + x_1 \frac{\partial p_1}{\partial x_1} + x_1 \frac{\partial p_1}{\partial x_2}\left(\frac{\delta x_2}{\delta x_1}\right),$$

from which we proceed much as before. This highly general solution can be applied whatever is the relation between the demands for the products; it can thus be applied to cases where the products are com-

plementary instead of competitive.[31] Here $\dfrac{\partial p_1}{\partial x_2}$ will probably be

positive, so that it is an anticipated consequential expansion of the other's output which will raise the marginal revenue curve of the first duopolist, and *vice versa*.[32]

III. BILATERAL MONOPOLY

"Bilateral Monopoly" is a phrase which has been applied to two different problems, and it is well to keep them distinct. The first is the case of isolated exchange, or of exchange between a group of buyers and a group of sellers, each acting in combination. Now so far as this problem is concerned, when the exchange is studied *in vacuo*, without reference to other people (outside the two groups) who may be indirectly concerned, I think one may say that there is complete agreement among economists. It has been evident since the days of Edgeworth that isolated exchange leads to "undecidable opposition of interests,"[33] and that therefore the problem is indeterminate, in the sense that the mere condition of each party seeking its maximum advantage is not sufficient to define an equilibrium.

The second problem is a more complex one. It arises when the commodity sold is a raw material or factor of production; so that we have also to take into account the relation of the buyer of the raw material to another market—that in which he sells his finished product. For this problem there existed a solution alternative to Edgeworth's, that

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February 1934. Mr. Nicoll's case of tacit combination emerges if we write

$$\frac{\delta x_2}{\delta x_1} = \frac{x_2}{x_1}, \quad \frac{\delta x_1}{\delta x_2} = \frac{x_1}{x_2}.$$

[31] Cf. Edgeworth, "The Pure Theory of Monopoly," *Papers* II, 122–126.

[32] See further, on the subject of duopoly, Professor Divisia's paper to the Leyden meeting of the Econometric Society, summarized in ECONOMETRICA, June 1934, and also in the *Revue d'Économie politique*, May 1934.

[33] *Mathematical Psychics*, p. 29.

of Cournot; Cournot had concluded that this more general problem is determinate. Here, as in the question of duopoly, Cournot has his modern followers; his position is defended by Dr. Schneider, and also, though with considerable qualifications, by Dr. Zeuthen.[34]

It must be confessed, however, that the reader of their works finds it very difficult to see just how the presence of a consumers' market makes any difference to the opposition of interests deduced by Edgeworth; and we have the authority of Professor Bowley in support of the view that there is indeterminateness also in the more general case.[35] Personally, I find myself in agreement with Professor Bowley; but I think it may be worth while to restate Professor Bowley's argument in terms of the *marginal revenue* concept, since this seems to make the crux of the dispute clearer than it has been made up to the present.
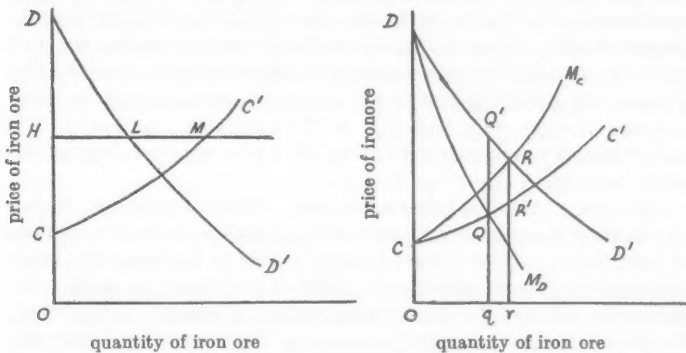
A, a monopolist producer of raw material (iron ore), is selling to B, a monopolist producer of finished product (steel). Now, as we have seen, B's demand curve for iron ore (*DD'*) is given by the marginal value product of iron ore (i.e., marginal physical product of iron ore in steel production × marginal revenue from the sale of steel); while A's supply curve of iron ore will be given by his ordinary marginal cost curve (*CC'*). That is to say, if a particular price OH is fixed by some external authority, A would be willing to supply HM units, B would be willing to take HL units; the amount actually sold will be whichever of these is the less. Now, within limits, the higher the price fixed, the greater will be A's profits, the lower the price fixed, the greater will be the profit of B. There is thus an opposition of interests. But this only within limits; for after a point it would not pay A to push up the price

[34] Schneider, *Reine Theorie*, ch. 2; Zeuthen, *Problems of Monopoly*, pp. 65 ff.
[35] "Bilateral Monopoly," *Econ. Jour.* 1928.

any further. The output which maximizes A's profits will be given by the intersection of the curve marginal to DD' with CC'. DD' is the demand curve confronting A; we can draw a marginal revenue curve (DM_D) corresponding to it, to cut CC' at Q. A vertical line through Q cuts the horizontal axis in q, and DD' in Q'. Then the most profitable position for A is when his output is Oq and his price $Q'q$.

If on the other hand, B can fix the price, what is the point where his profits are maximized? This is found by drawing a curve marginal to CC' (CM_C), to intersect DD' in R. Draw $RR'r$ perpendicular to the horizontal axis. The output most favourable to B will then be Or, and the price $R'r$.

Thus there does seem to be an "opposition of interests"; how did Cournot and his followers come to an opposite view? They would hold that there is an equilibrium with the price at $Q'q$, for in this case both producers are earning a maximum monopoly profit, B from the consumers of steel, A from B. That is perfectly true; no monopoly action by A can stop B earning a monopoly profit from the consumers. But A is not only a monopoly seller with regard to the consumers; he is also a monopoly buyer with respect to A. If he is allowed to do so, he will also extract a monopsony profit from A; it was this that Cournot left out of account.

As we have said, this indeterminateness does not mean that the law of causality is suspended; it only means that the static assumptions of fixed demand and cost curves do not suffice to determine the price. Attempts have been made by Dr. Zeuthen and myself to reach a determinate solution by introducing more "dynamic" factors.[36] Dr Zeuthen's solution proceeds by examining the probability of each side breaking off relations, which correspond to each set of terms; mine by considering the length of time for which either party would be willing to "strike" in order to get any particular price. The two methods appear to be complementary.

IV. CONCLUSION

I have so far confined my remarks to the purely formal aspect of recent work on monopoly; but in conclusion something ought to be said about the applicability of this now well-developed technique. It is evidently the opinion of some of the writers under discussion that the modern theory of monopoly is not only capable of throwing considerable light on the general principles underlying an individualistic eco-

[36] Zeuthen, *op. cit.* ch. 4; "du Monopole Bilatéral," *Revue d'Économie politique*, 1933; Hicks, *Theory of Wages*, ch. 7; A treatment somewhat similar to Dr. Zeuthen's is to be found in G. di Nardi, "L'Indeterminazione nel Monopolio bilaterale," *Archivo Scientifico*, Bari, 1934.

nomic structure, but that it is also capable of extensive use in the analysis of particular practical economic problems, that is to say, in applied economics. Personally, I cannot but feel sceptical about this.

We have already seen, in the case of duopoly, that the marginal revenue of a duopolist depends upon a term which can properly be called "conjectural." It is not the actual degree to which the second seller's output would change—it is the estimate of this degree on the part of the first seller. But once we have seen this, why mark this term only as conjectural? Is not the slope of the individual demand curve confronting a simple monopolist conjectural too? There does not seem to be any reason why a monopolist should not make a mistake in estimating the slope of the demand curve confronting him, and should maintain a certain output, thinking it was the position which maximized his profit, although he could actually have increased his profit by expanding or contracting.[37]

It is this subjective character of the individual demand curve which leads one to scepticism about the applicability of the apparatus. For what are the objective grounds from which we can deduce the existence of a significant degree of imperfect competition? It may be said that as soon as we find firms concerning themselves with a price policy, or undertaking selling costs, some degree of imperfect competition must be present. This may be granted;[38] but what degree? Is it important or negligible? There is no means of finding out but to ask the monopolist, and it will be kind of him to tell us.

Whether competition is perfect or imperfect, the expansion of the individual firm will be stopped by factors which are purely subjective estimates; in the one case by rising subjective costs or costs of organization;[39] in the other by an estimated downward slope of the marginal revenue curve. Objective facts give us no means of distinguishing between them.

The new theories seem to make little difference to the laws of change as they are exhibited in the traditional analysis; usually they do no more than suggest new reasons why we should get certain familiar effects, and there is very little means of distinguishing between the

[37] This argument is fortified if the demand curve is interpreted (as for most purposes it probably ought to be) as a fairly "long-period" demand curve.

[38] Professor W. H. Hutt, "Economic Method and the Concept of Competition," *Economic Journal of South Africa*, June 1934, disputes this as far as selling costs are concerned. His argument would appear to be valid so long as advertisement and product are sold in fixed proportions, but it ceases to be so if the "coefficients of consumption" are variable.

[39] Cf. E. A. G. Robinson, "The problem of management and the size of firms," *Econ. Jour.*, June 1934, and the same author's *Structure of Competitive Industry*. Also Kaldor, *op. cit.*

new reasons and the old. Whether an industry is monopolized, or duopolized, or polypolized, or operates under conditions of perfect competition, we shall expect a rise in demand to lead to a rise in output (though in all cases there are possible, but highly improbable, exceptions); and it is still likely that the rise in demand will be accompanied either by no change in price, or by a rise. New reasons are indeed adduced why a rise in output may be accompanied by a fall in price; it may be due to a rise in the elasticity of demand to the individual firm, rather than to economies of the Marshallian type. But the new explanation is not overwhelmingly convincing, and does not drive the Marshallian from the field.[40]

It does indeed now become possible that a rise in supply—if it takes the form of an influx of new firms—may actually lead to a rise in price, as would not be possible under perfect competition. Yet the conditions for this to happen, that the influx of firms should make the demand curve confronting each firm in the industry *less elastic*, is so peculiar, that it is hard to attach very much importance to this case—at least, as analyzed.

It is therefore hard to see that the new analysis does much to displace Marshallian methods. Marshall's assumptions are simpler, and if we are unable to tell which of two hypotheses is more appropriate, the simpler has the obvious claim to be chosen. But of course this is not to say that in strong cases—cases, for example, where discrimination is practiced—we are not obliged to assume monopoly conditions, and to make what use we can of the elaborations here described.

From this point of view, substantial gains have certainly been made; we are now in the possession of a much more complete theory of monopoly than was the case a very few years ago. If, when we have it, it seems less use than had been hoped, this is not an uncommon experience in the history of human thought.

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[40] It is tempting to propose a rehabilitation of Marshall on the basis of these recent developments. Since it has become clear that "increasing returns" are mainly a matter of indivisibilities and discontinuities, it is very possible that a firm may be in perfect competitive equilibrium with its (conjectured) demand curve horizontal, at the point of equilibrium, although it knows that a considerable increase in output would enable it to diminish average costs (of hired factors) considerably. But it is uncertain whether so large an increase in sales could be brought about without a considerable reduction in price, and refrains from expansion because it is unwilling to take the risk. This seems at least as plausible a construction as the other, and better suited to a world of very imperfect knowledge.

On the general question of discontinuity in cost, see M. Joseph, "A Discontinuous Cost Curve," *Econ. Jour.*, Sept. 1933.

ECONOMIC THEORY OF THE SHORTER WORK WEEK

By Charles F. Roos

In the United States there has been much agitation of late for a shorter work week. In the spring of 1933 Senator Black of Alabama introduced a bill in the United States Senate to limit the work week to thirty hours. This bill passed by a vote of 53 to 20, but was held up on a motion to reconsider. Simultaneously Representative Connery introduced a similar bill in the House. Support given to these bills undoubtedly assured the passage on June 13, 1933, of the National Industrial Recovery Act, one purpose of which was to increase purchasing power by decreasing working hours (thus spreading work) and by increasing wages. Decreasing hours without increasing hourly wage rates to compensate was impossible politically.

Prior to the Black and Connery Bills, the United States had been literally covered with the publicity and propaganda attendant upon the pronouncements of Technocracy. Although most of the claims of Technocracy have been proved to be false, the notion persists that great strides have been made in improving machines during the last decade, especially during the depression. In view of the debatable character of shorter work week proposals, it seems desirable at this time to examine some of the facts and to see how these are related to economic theory. In particular, it seems desirable to consider the economic experiment of the shorter work week carried out by the N.R.A., and to discover how hour shortening and attendant wage raising might lead to recovery or further depression.

Figure I shows how gainful workers by the different classes of workers varied from 1880 to 1930. In 1880 about 8 per cent of all gainfully employed were professional workers compared with about 25 per cent in manufacturing and mining. About 47 per cent were engaged in agriculture and fishing. Employment percentages, however, did not remain static. The change has been very considerable in the last thirty years. For example, percentage of those engaged in agriculture and fishing in 1930 was 23 per cent, compared with 47 per cent in 1880. In manufacturing and mining a slight decrease occurred over the previous ten-year period. Table IV shows changes in workers in manufacturing and man-hours required during the period 1920–1933. During the period of industrial expansion, we were opening new manufacturing industries and building up our own industrial plants. Improved manufacturing technique and things of that sort were displacing workers and thus fewer were needed in manufacturing, but employment in trade

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was going up at an accelerating rate and employment in the services was also increasing. In the period of expansion the United States had
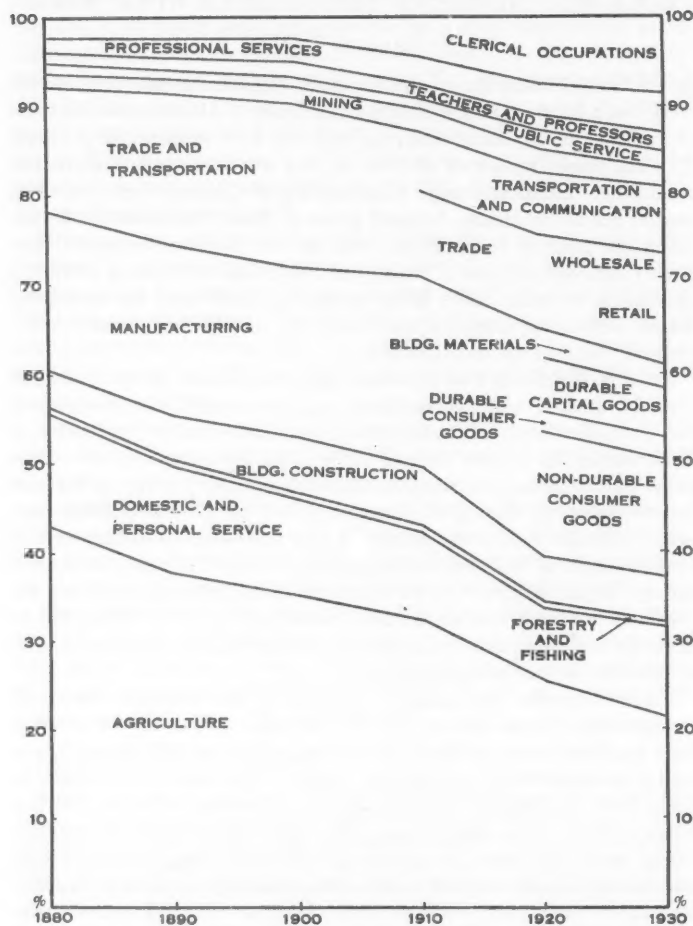
FIGURE 1.—Percentage distribution of gainful workers, by classifications.

something for everyone to do. A man might have to change his trade, but there was no question as to how long he could work.

Theoretically, it should be possible to have machines do most of the

manufacturing labor and allow human labor to be engaged almost exclusively in distribution, services, professions, etc., but then it would be necessary to insure that the machines operated. In such an economy however, cost and value would be quite different from their classical conceptions, since contributions of machines to value added by manufacture would be based on the "value" of the machines, which might be rather arbitrary and which would certainly depend upon the amount of product produced and sold. As long as individuals in such an economy were free to save or spend as they wished, economic conditions might be rather unstable. On the other hand, if every man now worked twelve hours a day, there would be no need for the movies. If man had no leisure, he would only need his automobile to go to and from work. A working man employed twelve to fifteen hours per day would need only food, clothing, and a comfortable bed. Thus, workers must have leisure to promote a consumption of certain goods and services.

If a man had no leisure in which to enjoy modern American civilization and merely devoted all his time to producing consumptive goods, there would have to be a very important change in total productivity. For example, the number of units per man-hour per workman produced in the boot and shoe industry increased about 50 per cent from 1923 to 1933, as shown in Table II. The same number of workmen could produce twice the number of shoes. This is partly due to better machinery but more to changes in the kinds of shoes made. In 1923 shoes were required to last for a considerable time, whereas today style is more important than durability. This change in consumptive habits explains a great part of the change in man-hour productivity. Less man-hours were needed to produce the same number of shoes, chiefly because the shoes were different.

The great increase in labor productivity mentioned in the previous paragraph has not occurred, because working hours have been gradually decreased. Consequently, employee productivity has not changed nearly so much as man-hour productivity has. Indices of employee productivity for five industries are shown in Table I.[1] There seems to be very little uniformity in the five industries with respect to employee productivity—a fact which is probably explicable on the basis of the different policies in different industries with regard to part-time employment. For the first six months of 1934 in comparison with the similar periods of 1933 and 1932, changes in employee productivity are shown in Table III. The percentage changes (although smaller) follow fairly closely the changes in man-hour productivity for the periods covered.

[1] This Table and Tables II and III were prepared by Emily Pixley as part of a report of the Research and Planning Division of the N.R.A.

It is usually assumed in economic theory that decreased cost leads to decreased price and hence to increased consumption. This hypothesis is, however, open to serious question. The United States, as a nation, would buy almost as many shoes at $5 as at $4, so that a twenty per cent decrease in price would make very little change in consumption. There are many products of this kind for which price increases

TABLE I

INDEXES OF EMPLOYEE PRODUCTIVITY

(1923–25 = 100)

Year	Slaughtering and meat packing	Boot and shoe	Rubber tire	Cement	Carpets and rugs
1920	73.3	—	—	80.4	—
1921	84.7	100.0	—	94.8	—
1922	96.0	102.6	—	84.5	—
1923	97.4	100.6	88.6	96.8	107.9
1924	103.7	98.3	108.9	99.0	93.6
1925	98.9	101.0	106.9	103.8	97.0
1926	98.7	103.0	107.4	107.7	90.0
1927	98.9	109.1	112.8	117.0	92.9
1928	101.3	119.8	120.3	127.4	100.2
1929	97.2	113.4	117.1	126.2	102.9
1930	97.7	102.4	120.4	127.7	84.9
1931	108.2	112.5	138.7	127.7	88.9
1932	111.5	116.4	128.2	113.6	82.7
1933	110.7	126.8	129.4	102.2	99.0

TABLE II·

INDEXES OF MAN-HOUR PRODUCTIVITY

(1923–25 = 100)

Year	Slaughtering and meat packing	Boot and shoe	Rubber tires and tubes
1920	75.3	—	—
1921	89.8	98.8	—
1922	96.9	100.2	—
1923	97.4	99.6	88.0
1924	103.5	100.7	104.7
1925	99.2	99.7	106.7
1926	98.5	104.3	106.2
1927	98.0	109.1	110.5
1928	99.3	115.2	123.8
1929	95.1	116.0	116.2
1930	97.1	114.6	129.4
1931	109.8	114.0	162.7
1932	115.0	128.0	172.2
1933	122.6	144.0	175.3

<div align="center">

TABLE III

PERCENTAGE CHANGES IN PRODUCTIVITY

January–June 1934 over January–June 1933 and January–June 1932

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	Per cent change 1934	Cement	Boot and shoe	Carpet and rug	Slaughtering and meat packing	Rubber tires
Over 1933	Employee productivity	7.1	−1.9	3.8	−14.8	−9.6
	Man-Hour productivity	13.1	8.5	10.4	2.0	−7.8
Over 1932	Employee productivity	4.5	14.8	28.0	−10.8	−17.1
	Man-Hour productivity	30.8	21.4	14.8	7.9	−7.5

do not greatly decrease the amount of products bought. Great saving in costs in such industries does not mean greatly increased production. Thus, it is necessary that the workmen displaced in the shoe industry find jobs in other industries. If they find other jobs, everything goes along beautifully. From 1920 to 1926, when a workman was displaced in any of the consumers' industries, he could almost always find a job in the construction industry or in distributing goods, so that an avenue of employment was open to the technologically displaced. This shift from manufacture to distribution has been going on for many years.

Furthermore, the population of the United States has been growing steadily for the last fifty years at a rate which meant that decreased cost due to machine installations produced sufficiently greater demand to make it unnecessary to displace the older workers. It was merely necessary to advise the youngsters not to enter the declining trades.

Whether or not decreasing cost by replacing workers by machines results in technological unemployment in an industry depends upon many factors. These include ratio of white collar workers and distributors to machine and factory employees, purchasing power, and demand for the product.[2]

[2] On page 180 of my book *Dynamic Economics* (Bloomington, 1934) the following statement appears:

"Suppose, however, for definiteness, that before mechanization . . . there are as many white collar workers and distributors as machine employees, and suppose further that through mechanization, r, the output per machine worker is increased 10 per cent. . . . if technological unemployment is not to occur, then

$$p_1 \leqq 1.04p(I/I_1) + .18b/aI_1$$

Since the production rate yielding maximum profit to the producer decreases as consumer income decreases,[3] it is very easy to see how unemployment in such industries as construction, coupled with displacement of several hundred thousand workers in manufacturing industries (see Table IV) can give the appearance of great technological unemployment. If hours are shortened over a period of time, for example, 1880 to 1930, by custom or otherwise, it is necessary to find something for workmen to do in their leisure time. Leisure in the United States must surely account in part for the great development of movies, theaters, schools, the automobile and radio industries, and many others.

In the case of the shoe industry, changes in man-hour productivity amounted to nearly 50 per cent, but, as indicated in Table II, employees' productivity only increased about 25 per cent; that is, the employees were working less hours in that industry. In the rubber tire and tube industry, man-hour productivity was almost doubled, but employees' productivity showed an increase of only about 40 per cent. A similar story can be told for the twenty industries whose statistics are available. Now the situation indicated shows that there would have to be an increase in demand for shoes of about 25 per cent or else there would be a decrease in the amount of labor employed in manufacturing shoes.

The general employment situation in the spring of 1934 was about as follows: There were ten million unemployed altogether. Of these ten million, approximately five million could be attributed to the building trades and to durable goods industries. An additional four million were idle in the service trades. Less than 800,000 were idle in consumptive goods industries. In other words, practically all the unemployment existed in the heavy goods industries and in the trade and service industries. All three of these groups had made rapid expansion in the period from 1923 to 1933, as indicated in Figure I.

Some groups believe that a solution of the problem of producing goods has been reached and that, since distribution economies are now

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where p is the price before mechanization, I is the purchasing power before mechanization, $y = ap + b$ is the demand for the product, p_1 is the price after mechanization and I_1 is the purchasing power after mechanization.

If $I_1 = I/2$, that is, if the consumer income is cut in half while the technological change is being made (not necessarily because of the change), the above expression becomes

$$p_1 \leq .52 - .18(b/|a|)I_1$$

where $|a|$ denotes numerical value of a, which means that *the new price must be less than half the price before the technological change if there is to be no technological unemployment.*"

[3] C. F. Roos, *op. cit.*, pp. 167–170 and 69–110.

TABLE IV
MANUFACTURING INDUSTRY
Production, Employment, and Man-Hours

Year and quarters	F.R.B. Index of mfg. production (unadjusted)	Average number employed (thousands)	Average hours per week per worker	Total man-hours per month (millions)
1920—1 (peak)	95.3	10,155	50.0 E	2,204
1923—2 (peak)	107.0	9,387	49.9	2,037
1926—1 (peak)	110.3	9,057	48.6	1,910
1929—3 (peak)	121.7	9,242	48.5	1,950
1930—1	107	8,295	46.2	1,660
2	104.7	8,078	45.1	1,580
3	89	7,624	42.4	1,405
4	81	7,242	41.7	1,310
1931—1	87.7	6,877	41.6	1,240
2	81.3	6,823	42.0	1,240
3	77.3	6,607	39.8	1,140
4	69.3	6,171	38.1	1,020
1932—1	68.7	5,931	36.6	940
2	60.7	5,474	33.6	800
3	59.7	5,260	33.2	750
4	62.3	5,414	36.1	850
1933—January	63	5,171	37.4	839
April	68	5,144	38.0	848
July	97	6,140	42.3	1,127
October	77	6,509	35.7	1,008
1934—January	75	6,054	33.7	885
April	89	7,068	36.2	1,110
July	71	6,749	33.4	978
October	80 E	7,128 E	34.0 E	1,052

Prepared by Victor Von Szeliski and Alexander Sachs.
E—estimate.
Employment calculated from Federal Reserve Board unadjusted index of employment in manufacturing industries, by multiplying by 89.
Hours per week from National Industrial Conference Board *Service Letter*. United States Bureau of Labor Statistics has a service, probably more comprehensive than that of the N.I.C.B., starting October 1933. This is presumably based on a larger sample than the former.
Monthly hours equal number employed $\times$ weekly hours $\times$ 4.34, there being that many weeks in a month on the average.
Federal Reserve Board Index is adjusted for working days but not for seasonal.

inevitable, it is necessary to shorten the work week drastically. This assumption is based largely on the fact that these believe that the

United States has enough buildings and machinery for a while and that it has too many hotels and restaurants. They feel that displacement in the building industries, trade, and services can only be rectified by the shorter work week. In other words, since workmen have to be put somewhere, why not put them in manufacturing and distribution by a mandatory short work week?

Acting on this belief, pressure for a shorter work week was brought to bear by the United States Secretary of Labor and by many labor leaders. Many governmental advisers have said, and are saying, that the United States must work towards a shorter week. Some advisers, however, including A. J. Hettinger, George F. Warren, and the author, believed that drastic shortening of the work week would lead to decreased production. These were not opposed to shorter work weeks in principle, but failed to see how increased costs of building, machinery manufacturing, furniture, and automobiles, would lead to increased demand and consequent increased employment. In other words, these felt that the shorter work week in the heavy industries cannot increase employment, since these industries rely on the investment of savings and must offer adequate profit incentives.

Suppose that the working hours and payrolls of the 100,000 workmen of an industry (not a plant) are as given in Table V.

TABLE V

HYPOTHETICAL FIGURES IN ACCORDANCE WITH FACTS NOT AVAILABLE FOR PUBLICATION

Hours of work per week	Number of workmen	Total hours per week	Weekly payroll (dollars)
5–10	2,000	15,150	7,500
10–15	3,000	37,440	18,750
15–20	8,000	141,010	70,000
20–25	11,000	247,520	123,750
25–30	14,000	385,110	179,950
30–35	22,500	715,020	393,250
35–40	20,500	749,780	412,500
40–45	12,000	509,970	265,200
45–50	4,000	190,340	85,500
50–55	2,000	105,290	42,000
55–60	1,000	57,330	23,000
Totals	100,000	3,153,960	1,621,000

From the above table it is seen that the average work week was 31.54 hours in this industry. Assuming no decrease in demand due to increased costs, a maximum work week of 40 hours would result in re-employment. The total number of employees working 40 hours and

over is the sum of the last four figures of column 2, that is, 19,000. If each of these worked 40 hours, the total hours of this group would be $40 \times 19,000 = 760,000$, as against the 862,930 given above. In other words at least $862,930 - 760,000 = 102,930$ hours would be available for distribution among new employees. Thus, if each of the new employees worked 40 hours, this industry would employ at least 2,573 additional workmen. The forty-hour maximum would not adversely affect those plants which were working less than forty hours. Plants working more than 40 hours would either spread work and keep their sales or let part of their sales go and spread work only partially. In the latter event, plants working part time would probably show increased hours and increased sales. Spreading work by law may, therefore, result in increased costs which are greater than the wage increases.

Of the seventy-eight industries for which data are available, the thirty-hour week in March, 1934, would have absorbed the unemployed in only twenty-nine industries, assuming no decrease in demand. For the other forty-nine industries, work weeks varying all the way from 30 hours to 6 hours would be required. If the shorter work week plan should be applied to the industries which had the greatest amount of unemployment, that is, the heavy goods industries, it probably would increase the unemployment, since demand would be considerably lessened because of greatly increased costs. There would be less work to divide up than previously and consequently standards of living would be lowered. Demand is determined differently in the consumers' goods industry from the way it is determined in the capital goods industries. For the capital goods industries the ratio of expected income to replacement cost is the factor which is important in determining demand.

For a long time the Administration appeared to have the idea that all industries could be treated almost alike as regards hours and wages. Reports which seemed to call for decreased hours and increased pay in the durable goods industries were released by the Department of Labor during code negotiations designed to decrease hourly rates in the durable goods industries in exchange for guarantees of more work. The Administration finally became conscious of differences between capital goods and consumers' goods, and during the code authority conferences in March, 1934, General Johnson ordered industries to be separated into three classes. In pursuance of his order the author and Mr. John Hamm of the Research and Planning Division divided industry into durable and non-durable goods and services. There were, of course, many division line products. The automobile might be either a durable or a non-durable good, depending upon the criteria of division. Hence, after the division, many industries were not sure that they were in the right group. The automobile industry raised an exception and said that

it was not a durable goods industry. It had been placed in the durable goods industry group because it was believed that automobile demand was sensitive to price changes.

Economic theory has failed to take into account adequately the relation of man to the machine. Theorists have generally very glibly referred to the supply of labor without specifying the length of the work week, whereas obviously, if all workers were employed on a forty-hour average work week, some would be unemployed if the work week could be suddenly increased to fifty hours.[4]

What about the work week? Obviously a way must be found to regulate machines and jobs, either by speeding up consumption whenever it slackens or by artificially strengthening the bargaining power of the workers. As I have said previously, a bill for the thirty-hour work week was sent to Congress in 1933. Table IV indicates that in 1931 the thirty-hour work week law, with flexible provisions for overtime to avoid bottlenecks caused by shortage of skilled labor, might have prevented a severe depression, but such a law would not work in 1933, when there was so much unemployment. Even Arthur Dahlberg, one of the chief proponents of the shorter work week, reported to General Johnson in March 1934, that a 10 per cent decrease in hours of labor and a 10 per cent increase in wages was not advisable at that time. The chief reasons given were that industry had not yet made adjustments to the previously shortened work week brought about by codes, and that it was academic to shorten the work week more until compliance with the reductions already made was secured.

During the Code Authority conferences many industrialists complained of shortages of skilled workers. The problem often comes up that there may be a job in Maine and the skilled worker fitted for the job is in California or Texas. Many times it is not feasible to transport a worker across a continent in order to give him a job which may last only a month. Then there is the problem of housing. In small towns drastic shortening of the work week may result in greatly decreased production, at least for several months or a year, due to lack of housing for workers who have to be imported. The problem is not so acute in the large cities, since there is normally a greater housing vacancy there and skilled workers both come and go. In any event, however, important shifts in population and markets must be expected to accompany mandatory changes in the work week. Also, shortage of skilled workers necessarily means more unemployment for the unskilled workman, since the latter cannot work until the former has laid out his work.

Even a skilled worker must learn the methods of his factory before

[4] Arthur Dahlberg considers this problem in his book, *Jobs Machines, and Capitalism*, New York, 1930.

he really earns his wages. Thus, even the most skillful printer requires about a week to familiarize himself with printing technique and shop rules. A shorter work week resulting in the hiring of additional hands must of necessity result in increase of costs by more than the wages paid to the extra help, unless freedom from fatigue results in greater man-hour productivity. In the United States the work week was perhaps sufficiently short in early 1933 to render the fatigue element rather unimportant.

One of the best examples of important difficulties due to skilled worker bottlenecks is given by the ship building industries. Here only certain skilled workers are needed in the early days of construction. Hence delaying the early construction work by limiting the time of skilled workers in the early stages of keel laying, etc., postpones the time at which unskilled workers can be employed.

TABLE VI
RATIO OF WAGES TO VALUE ADDED
Frequency Distributions for Industries

Ratio wages to value added	Non-durable consumer goods	Durable consumer goods	Capital goods
.00–.04	2		
.05–.09	2		
.10–.14	7		
.15–.19	9	1	
.20–.24	5	1	2
.25–.29	7	3	5
.30–.34	10	4	4
.35–.39	14	12	9
.40–.44	10	9	14
.45–.49	7	6	8
.50–.54	3	1	5
.55–.59	1		1
.60–.64			2
Total	75	37	48

It should also be pointed out that all industry does not have the same ratio of wages to value added to the products by manufacture. (See Table VI.) A shorter work week hurts some industries and benefits others. Furthermore, a study of Table VII shows that states would receive unequal treatment in every industry if a mandatory shorter work week were imposed. The industry which has a low ratio of wages to value added does not need to make as great an increase in wages as the one having a high ratio.

Shortening of the work week tends to unbalance industry. The short work week must of necessity give first stimulus to the common con-

TABLE VII

RATIO BY STATES

A — $\dfrac{\text{Wages}}{\text{Value added by manufacture}}$

B — $\dfrac{\text{Cost of materials}}{\text{Value of product}}$

Non-durable consumer goods

	Printing and pub. news and periodicals		Clothing, women's		Clothing, men, youth, boys		Paper and pulp		Knit goods		Cotton goods		Worsted goods		Boot and shoe other than rubber		Rubber tires and inner tubes		Silk and rayon manufactures	
	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
NORTH ATLANTIC STATES																				
Delaware	.32	.23	.25	.27	.17	.55	.60	.72	.57	.54	.23	.51	.58	.64						
Connecticut	.17	.19	.55	.46	.49	.29	.30	.47	.47	.48	.64	.58	.48	.48	.51	.53	.30	.59	.48	.51
Maine	.25	.23	.49	.54	.50	.52	.33	.63	.57	.54	.65	.57	.52	.62	.55	.51				
Massachusetts	.24	.27	.39	.54	.44	.52	.37	.53	.43	.50	.61	.54	.58	.66	.54	.58	.30	.59	.43	.56
New Hampshire	.31	.16	.49	.54	.65	.64	.41	.56	.51	.52	.60	.58	.58	.55	.45	.51				
New Jersey	.28	.20	.52	.41	.57	.43	.36	.56	.55	.50	.64	.53	.50	.64	.51	.49	.57	.60	.49	.55
New York	.14	.21	.28	.55	.32	.49	.39	.63	.43	.53	.51	.50	.61	.64	.50	.51	.30	.59	.34	.62
Pennsylvania	.16	.25	.38	.53	.43	.51	.32	.57	.53	.49	.47	.51	.43	.66			.32	.55	.40	.54
Rhode Island	.25	.21	.49	.54	.17	.64	.37	.49	.51	.65	.52	.52	.49	.66					.58	.58
Vermont	.29	.15	.33	.33	.17	.64	.29	.57	.41	.46	.65	.57	.58	.66						
MIDWEST STATES																				
Indiana	.22	.21	.42	.52	.47	.44	.39	.61	.41	.43	.63	.66	.58	.66	.38	.54	.44	.60		
Illinois	.18	.23	.42	.53	.41	.41	.36	.58	.46	.51	.52	.59			.39	.53	.44	.60		
Iowa	.20	.20	.11	.49	.36	.58	.33	.56	.29	.41					.40	.56	.45	.67		
Kansas	.23	.20	.63	.48	.55	.55	.33	.56							.36	.40				
Michigan	.18	.23	.40	.56	.41	.49	.41	.59	.40	.46	.26	.59			.67	.71	.44	.60		
Minnesota	.20	.23	.35	.56	.51	.58	.30	.60	.32	.42					.42	.53	.44	.60		
Missouri	.22	.26	.46	.57	.45	.54	.33	.56	.57	.65	.36	.60			.44	.61	.42	.52		
Nebraska	.24	.21	.36	.58	.17	.64											.42	.52		
North Dakota	.27	.16																		
Ohio	.19	.25	.34	.50	.51	.53	.41	.60	.46	.56	.23	.51	.58	.66	.51	.53	.38	.54		
Oklahoma	.25	.22	.36	.58							.36	.60								
South Dakota	.27	.19			.17	.64														
West Virginia	.28	.18	.43	.52	.17	.55	.36	.62	.57	.54	.23	.51	.30	.65	.45	.55				
Wisconsin	.21	.21	.41	.53	.49	.57	.34	.61	.40	.41			.58	.66			.28	.55		

RATIO BY STATES

A — Wages / Value added by manufacture
B — Cost of materials / Value of product

Non-durable consumer goods

	Printing and pub. news and periodicals		Clothing, women's		Clothing, men, youth boys		Paper and pulp		Knit goods		Cotton goods		Worsted goods		Boot and shoe other than rubber		Rubber tires and inner tubes		Silk and rayon manufactures	
	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
SOUTHERN STATES																				
Alabama	.20	.22	.27	.38	.71	.41	.22	.54	.50	.59	.45	.63					.32	.65		
Arkansas	.26	.21	.46	.54			.22	.54	.57	.65	.53	.70								
Dist. of Columbia	.15	.22	.25	.27			.37	.49												
Florida	.27	.19			.65	.22	.37	.49												
Georgia	.21	.21	.37	.54	.27	.52			.43	.56	.46	.64	.30	.65		.56	.32	.65		
Kentucky	.25	.26	.25	.27	.46	.51	.24	.63	.49	.51	.63	.70	.30	.65	.51					
Louisiana	.21	.25	.28	.58	.38	.52	.53	.73	.57	.65	.36	.60				.54	.30	.59	.55	.27
Maryland	.24	.27	.39	.51	.38	.48	.22	.54	.42	.51	.47	.57	.30		.51				.37	.59
Mississippi	.28	.19	.44	.64			.26	.67	.57	.54	.47	.61					.32	.65		
North Carolina	.22	.19	.27	.38	.71	.41	.37	.49	.48	.56	.49	.61	.30	.65						
South Carolina	.24	.20	.27	.38			.35	.57	.53	.58	.55	.64								
Tennessee	.23	.23	.41	.47	.46	.52	.22	.54	.52	.55	.45	.61	.30	.65	.34	.57			.48	.44
Texas	.23	.24	.38	.49	.45	.41	.34	.68	.57	.65			.30	.65	.33	.39				
Virginia	.23	.21	.41	.58	.45	.58			.45	.56	.50	.54	.30	.65			.32	.65	.39	.55
ROCKY MOUNTAIN STATES																				
Arizona	.21	.17																		
Colorado	.19	.21	.34	.49	.50	.50	.33	.56									.42	.52		
Montana	.38	.24																		
New Mexico	.32	.19							.53	.52										
Utah	.23	.24	.36	.58					.29	.42										
Wyoming	.32	.16																		
PACIFIC STATES																				
California	.14	.22	.46	.53	.53	.45	.34	.54	.36	.44	.51	.62	.58	.66	.64	.48	.36	.60		
Idaho	.31	.17																		
Nevada	.28	.14																		
Oregon	.24	.22	.41	.55	.43	.51	.22	.52	.29	.41			.58	.66						
Washington	.20	.19	.41	.50	.51	.43	.29	.56	.57	.52					.42	.54				

sumptive goods such as food, cheap clothing, etc. Benefits due to increased employment are necessarily delayed for the heavy industries.

Many of the codes provided for forty-hour weeks, some for forty-four hours, and a few for forty-eight hours per week. In nearly all codes there are exceptions. Some provide exceptions for emergency repair crews. For example, in the Ship Building Code, these can work from fifty to sixty hours in an emergency. The code provides that the men can work forty-eight hours in building commercial ships, but in the government yards and in the construction of government ships the workmen can only work thirty-two-hours a week. This condition was almost made mandatory by the act which specified thirty hours for public works.

Many industries writing codes realized that they could not do business with rigid maximum work weeks, and some of these provided that the workman should receive extra pay for all over-time, except possibly during seasonal operations. From this point of view the automobile industry code is one of the most interesting. In the fall of the year new tools are being made and installed, and during the period of tooling there is, of course, very little manufacturing. The industry runs into a shortage of skilled manufacturing workers in the spring when it goes back to full production. The code, therefore, provides for longer hours during the period of peak production.

Exceptions for workers in the research and experimental fields were also often made.

Some codes, such as the one for the plate glass industry, provide for the exemption of certain kinds of skilled workers, that is, those whose number is definitely less than would be needed under the shorter work week. These are invariably in trades requiring long apprenticeship. Still another kind of exception occurred in the continuous process industry, that is, an industry for which the plant is run all the time.

The fact that industry has found it essential to put so many exceptions into the codes indicates that both the Black Bill in the U.S. Senate and the Connery Bill in the U.S. House are impossible and that, if they were enacted into law, their failure to recognize fundamental differences in industry would very soon make them unworkable. Nevertheless, a still shorter work week will come eventually, but it can best be accomplished gradually. As already pointed out, through the shorter work week a nation is able to use additional products.

It is interesting to examine the possible theoretical effects of a shorter work week on money, "values," and general well being. To facilitate such consideration it is desirable to write the equation of exchange in a capitalistic economy, as I have already done.[5] This takes the form

[5] C. F. Roos, *op. cit.*, pp. 233–245.

$$[MV + M_c V_c + M_b V_b]T = \sum_{i=1}^{F} p_{mi} q_{mi} + \sum_{i=1}^{G} p_{oi} q_{oi}$$

$$+ \sum_{i=1}^{C} p_{ci} q_{ci} + \sum_{i=1}^{R} p_{ri} q_{ri} + \sum_{i=1}^{S} p_{si} q_{si}$$

where

M = average amount of currency (gold, silver, paper notes) in circulation in the interval of time t to $t+T$;

V = average number of times this currency changes hands or circulates in the time interval t to $t+T$;

M_c average amount of bank credits subject to check in the interval of time t to $t+T$;

V_c = average velocity of circulation of M_c; and

F, G, C, R, and S are the number of transactions involving raw materials, consumer manufactured goods, capital goods, rents and interest, and wages or salaries, respectively, in the time t to $t+T$; and the p's refer to prices, and the q's to quantities of goods sold.

Prices of capital goods depend upon expected income and only slightly upon wages and material expended upon them. Thus, the prices p_{ci} are of the form: sum over some interval of time (sometimes regarded as such a long time that it can be an infinite interval of time) of $(pq - Q)$ where p represents price, q represents quantity of a good or service produced by the capital goods, and Q represents the cost of production of q, multiplied by a discount factor plus a fraction of the cost of production of the capital goods, that is,

$$p_{ci} = \sum_{i=1}^{N} \int_{t}^{t+t_1} [\gamma_{ij} p_{oi} q_{oi} - C(q_{oi})] e^{-\delta_i t} dt + \mu_i L_i,$$

where L_i is the construction cost of the capital goods; N is the number of products or services to be produced by the capital goods; $C(q_{oi})$ is the cost of producing the jth good or service (if a subscript s is used to denotes a service); δ_i is a discount factor (force of interest); γ_{ij} is a factor used in estimating future income, and μ_i is a mortgage or bond ratio, for example, fifty or sixty per cent for real property. The quantities γ_{ij} and μ_i are largely psychological, of course, and the former is more subject to fluctuation than the latter. This formulation of price applies particularly to plants. Some unassembled capital goods, of course, have their prices determined much like the prices of consumer goods.[6]

The price p_{gk} of a semi-finished consumer goods, i.e., a good in the first stage of processing, may be expressed by a relation

$$p_{gk}^{(1)} = \gamma_1 \left[\sum_{i=1}^{M} p_{mk} + \sum_{i=1}^{W_i} W_j^{(1)} \right],$$

[6] See C. F. Roos, *op. cit.*, pages 204–245.

where $W_j^{(1)}$ stands for services of capital (interest) and services of labor, including direct labor to produce power. For the second state of manufacturing

$$p_{gk}{}^{(2)} = \gamma_1 \left[\sum_{k=1}^{M_2} p_{gk}{}^{(1)} + \sum_{j=1}^{W_2} W_j{}^{(2)} \right]$$

and so forth for other stages of processing. Transportation and selling expense can also be included by proper modification of the definitions of W and p.

If the time of manufacture extends from $t-t_i$ to t the price of the consumer good may for simplicity be taken to be

$$p_{gi} = \int_{t-t_i}^{t} \left[\sum_{k=1} \lambda_{1k}(x, t) W_k(x) + \sum_{j=1} \lambda_{2j}(x, t) p_{mj}(x) \right] dx$$

where $\lambda_{ik}(x, t)$ is a psychological factor depending upon business conditions at each time x in the interval $t-t_i \leq x \leq t$ as well as upon business conditions at the time t. This formulation indicates how a change in the price of a raw material may be reflected through different stages of processing and distribution.

The ways in which other prices are determined need not concern us here.

Economists have rather generally condemned the National Recovery Administration as a recovery movement. Considerable criticism was in order, but most of the condemnations made have been pure rantings about something which has not been understood. Thus, the usual argument has been that it is unorthodox to raise costs before profits are raised. On the other hand an examination of the equation of exchange for a capitalistic economy indicates that there are several ways in which the N.R.A. might have contributed to recovery. Briefly these are:

1. Increased wage costs might cause the consumer to expect increased prices. In the uncertainty regarding what increases are justified, the entrepreneur may in general be able to recover more than his increased costs and make profits. Studies of price increases under the N.R.A. indicate that this happened. Furthermore, with prices increasing, purchases for consumption are speeded up and, since short time consumptive goods are quickly used up, a net gain in production might be expected. Reactions due to inventories must, of course, be expected. That these did occur is indicated by an examination of the Federal Reserve Board indices of production and inventories.

2. A fairly general increase in wage costs brings about increased values of inventories, that is, a reflation of values. When inventories

increase in value, corporations can hardly be prevented from making windfall "profits." Working capital position might be improved if inventories were financed by bank loans. Profits on management or inventories cause reflation of equity values, and increased equities ought to bring about a feeling of financial security to middle class holders of equities who, as a consequence, might spend more freely.

3. By increasing prices of goods and wages, corporation debts and taxes are effectively scaled down. Slow circulating income, that derived from interest, becomes a smaller portion of the national income.

4. Distribution of work without decrease in wages produces more consumers who are not living on savings or gifts, i.e., consumers who consume more. Since "fixed" costs change little, increased prices of consumers' goods do not defeat the whole program. Thus, a general increase in wage costs of ten per cent need not produce a ten per cent increase in price, since there are fixed prices not affected to any great extent by the N.R.A. As pointed out above, however, prices were increased more than the wage increases.

5. Distribution of work leads to an absorption of vacant housing through increased number of marriages and undoubling of families. This should shortly lead to greatly increased demand for residential construction. This demand, of course, will not develop until a considerable part of the present vacancy is absorbed, and will be considerably delayed by high construction costs, foolishly initiated by the Administration. Ratio of rent minus taxes to replacement cost is the important factor here. Expectation of higher rent will occur when vacancy is absorbed. This incentive factor has been largely nullified by increased building costs.

6. Expectation of profits is necessary in consumer goods industries before demand for capital goods, other than residential building, will develop.

7. Inventories provide the only check on runaway prices. Therefore, to accomplish most, higher wages and costs in consumer goods industries should be threatened for several months before being brought into effect. Once they are made effective, decrease in production for a short time can be expected, due to the pressure of inventories on high prices. After allowing several months for adjustment, increased costs should again be threatened and so on. This is really the difficulty of the problem. While hours are being shortened a bit, business is upset and a drop in production is almost inevitable. It is this factor which makes the author wonder whether any recovery at all can be obtained through the N.R.A.

In order that I be not misunderstood, I want to point out again that the preceding discussion of the N.R.A. has dealt only with some

theoretical aspects of the shorter work week and concomitant increases in pay forced by political considerations. In no sense of the word, therefore, is it to be regarded as an evaluation of the N.R.A., since all codes contained fair trade practice provisions in addition to wage and hour agreements. These additional aspects of the N.R.A. make it virtually impossible to evaluate the shorter work week aspects. It is entirely possible that greater reemployment would have occurred without the N.R.A. than with it, but my study of factors influencing residential building[7] does not indicate that this would have been the case, since building could not get under way in the period which has elapsed since the beginning of the N.R.A., and production of consumers' goods would have had almost to double to get the reemployment which has occurred.

The stimulating effect of higher wages and expected higher prices can be easily overestimated. Distributors of consumer goods undoubtedly increase their inventories when prices are expected to increase, thus leading to an increase in employment during this period, but when prices cease to advance orders drop below "normal" while the inventories are being liquidated. Since steadily rising prices can be attained only by disturbing business so that it will not enter into contracts, it is almost impossible to keep up this stimulation.

It is impossible to measure the "recovery" which is due to the N.R.A., since it is difficult to imagine that all the huge sum distributed by the Relief Administration has gone to persons who would otherwise have lived off savings, and the Public Works Administration must surely have resulted in some reemployment. Other factors having either positive or negative influence on "recovery" include the Banking Act (negative), the Securities Act (decidedly negative), the Civilian Conservation Corps (positive), monetary policy (probably positive although uncertainty must surely have prevented long time commitments), the Agricultural Adjustment Administration (zero or negative if regimentation features are properly evaluated), Tennessee Valley Authority (negative, since security deflation means destruction of potential purchasing power), Federal Housing Administration (probably negative since without it or Public Works desirable reduction in building costs would almost surely have come about) and the Home Owners Loan Corporation (probably positive).

By recovery it must be meant that the standards of living of the American people are raised by production of more goods and services. In other words the important question is whether Americans received more income in the form of new goods or use of existing idle goods such as housing. They have certainly used more housing, since vacancy has

[7] C. F. Roos, *op. cit.*, page 106.

decreased from about 14 per cent to about 5 per cent. Production of consumers' goods has fallen off but retail sales (presumably out of inventories) have increased in dollar value. No index of physical quantity is available, but I personally feel that this was less in 1934 than in 1933.

Some information can be obtained from changes in cash income of various groups. The following table represents my estimates of cash income in June 1933 and June 1934 in millions of dollars using the method of estimating developed by Victor von Szeliski.

TABLE VIII

CASH INCOME OF INDIVIDUALS BY TYPES
(Millions of Dollars Adjusted for Seasonal Variation)

	June 1933	June 1934	Per Cent Increase
Salaries, heavy industries	232	270	16.4
Wages, heavy industries	548	766	39.8
Salaries and Wages, other industry	1079	1270	17.7
Federal Salaries and Pensions	141	160	13.5
Public Education	135	110	−19.0
State and local Government	127	127	—
Total Labor Income	2262	2703	19.5
Dividends	150	171	14.0
Interest	436	413	−5.0
Rents and Royalties	73	93	27.4
Entrepreneurs	385	407	5.7
Farm Cash Income	583	550	−5.7
Farm Benefit	—	28	—
C.W.A.	—	11	—
C.C.C.	9	13	44.8
Total Income	3898	4389	12.6
Cost of Living, N.I.C.B.	72.8	78.8	8.2

This table shows that very little if any increase in purchasing power has occurred. The table, of course, includes effects of all government policies and not the N.R.A. alone.

It is possible the N.R.A. might have been regarded as an inflationary mechanism that could bring about a controlled inflation of prices which are most out of line. As administered, the N.R.A. consisted mostly of wishful thinking. It reminded me of "a ship without a rudder in a night without a star." Since the same treatment was applied to durable and non-durable goods and services, one may well suppose that things done just about cancelled each other. The final net *recovery* gain seems to have been an absorption of housing vacancy.

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COMPETITION UNDER SECRET AND OPEN PRICES

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I. INTRODUCTION

THE problem of "open prices" has been under discussion in the United States for twenty years, and more than ever since the passage of the National Industrial Recovery Act (N.I.R.A.). Up to the present, business men have displayed a greater interest in the question than have economists and statisticians. The principal issue involved is whether or not sellers of goods and services should be required to make their prices known to their competitors. Related questions—e.g., publicity to customers—have received less notice, partly because business men are naturally less interested in them. Normally a seller is only too willing to let customers or prospective customers know his prices; but he may fear that giving the same knowledge to competitors will simply enable the latter to take advantage of him. He himself, on the other hand, is anxious to know what others are charging. If each agrees to give up his own secrets for the sake of discovering those of the rest, the natural solution is to exchange prices through a central clearing-house. Such a clearing-house is known as an "open price association."

There was already an active trade association movement in American industry before the particular type known as the open price association came on the scene. These early trade associations were principally engaged in performing various services for their members, but sometimes helped them fix prices or curtail output in common. Beginning about 1911 some of the existing groups, as well as new ones, assumed the function of collecting from their members, and redistributing to them, the price and other statistical information developed within their industries. This step was taken largely under the influence of Arthur Jerome Eddy, a Chicago attorney, whose book, *The New Competition*, published in 1912,[1] emphasized the value of open prices in making competition more intelligent. The numerical progress of the movement is indicated by two surveys made by the Federal Trade Commission:[2]

	1921	1927
Number of open price associations	150	90
Number of statistical associations	474	256
Total number of trade associations	1273	860

[1] New York: D. Appleton & Co; revised edition, Chicago: A. C. McClurg & Co., 1915.

[2] Open-Price Trade Associations, Senate Document No. 226, 70th Congress, 2nd Session, Washington: Government Printing Office, 1929, pp. xvi–xvii, 14–15, 33, 35, 40.

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It is apparent from these figures that only about one-third of the associations which collect statistics on their trades have been gathering prices specifically, and that only about one-third of trade associations in general have collected statistics at all. Regardless of the fact that open price activity is almost always found in conjunction with other statistical work, and that statistical activity is almost always found in conjunction with other trade association work, this paper will ignore these other functions and analyze the open price activity alone.

According to the figures presented by the Trade Commission the number of trade associations of all sorts decreased during the twenties. The decrease shown is partly due to smaller coverage by the 1927 questionnaire, but it appears that a real decline did take place. In other words there was a tendency for less statistical work to be done. One reason for the decline in activity of trade associations was the legal tangle in which price and statistical reporting became involved after 1920.[3]

II. THE RÔLE OF OPEN PRICES IN ECONOMIC THEORY

From the standpoint of pure theory, open prices are essential to the proper functioning of the economic system, regardless of how the system is organized. It may be demonstrated that a competitive system requires open prices for its best operation.

Some have thought that the "New Deal" and, in particular, the N.I.R.A., signalized the end of free competition in the United States, but this view appears untenable. The prevention of destructive price cutting in the codes of fair competition has in only the minority of cases resulted in the transfer of the essential competitive acts of price and production determination from the individual producer to a central body. Moreover, the restoration of free competition where it has been imperiled (except in public utilities and natural resources) is the expressed policy of the leaders in the Administration. A discussion of open prices, therefore, must assume the competitive system.

This same assumption of free competition has been central to the whole structure of economic theory built up since the eighteenth century, and the economists who have contributed to it have stated or taken for granted that free competition requires a full knowledge by the participating buyers and sellers of all factors affecting their interests.[4] It is as fully understood that perfect knowledge is essential

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[3] The legal status of the trade association is discussed briefly in Appendix I.

[4] C. F. Roos and Henry Pixley have developed a theory of "loss leader" merchandising which assumes partial ignorance on the part of the consumer. See Chapter VII of Roos, *Dynamic Economics*, 1934, Monograph of the Cowles Commission for Research in Economics.

to perfect competition as it is that neither of these perfections is ever found. Market prices are one important aspect of economic knowledge, and they must be known if competition is to fulfill its purposes: (1) capital and labor will not be attracted into production of the goods most desired unless it is known what consumers are bidding highest to obtain; (2) the task of production will not be given to the most efficient producers if unethical rivals can take the trade by granting secret rebates; (3) the right amounts of different goods will not be produced if prices do not give the signal; and (4) the benefits of efficiency may not be passed on to the consumer if efficient producers do not even know that competitors are underselling them. In short, the price system is essential to the fulfillment of these purposes, and open prices are one aid to the free functioning of the price system.

The mere definition of competition makes it evident that open prices are of its essence. The effort of two or more sellers to dispose of their products or services to the same buyer by the offer of inducements to him necessarily involves a knowledge by each of what the others are offering. It would, indeed, be possible for a producer to ignore the others' terms and exert himself to offer all that he possibly can, in the certainty that, if any one else outbids him, he would have been unable to meet the bid even had he known of it. This would not, however, be competition between two producers, but a test of the willingness to suffer loss on the part of one producer. An analogy sometimes used is that of two runners. Unless each knew where the other was, he might speed to the point of exhaustion, but it would be a race against time or against breakdown rather than against a competitor.

Open prices facilitate competition among the purchasers as well as the sellers. When terms are secret, mass distributors can use their buying power as a club to obtain discounts even greater than those justified by the saving resulting from their quantity purchases. They are thus able to undersell their smaller competitors and ultimately drive them out of business. Open prices discourage this process, since a seller is naturally reluctant to grant discriminatory discounts to one customer if the rest are going to hear about it, either directly or through his own competitors. Sometimes, in fact, the remedy is excessive, sellers not daring to give big buyers even the discounts to which they are entitled by such factors as the saving in credit and sales expense, the placing of orders for manufacture in a normally dull season, etc. In many lines these factors are different for each sale, and a uniform schedule of open prices will hardly cover the matter. There is little wonder that the large distributors have been among the most determined opponents of the open price policy in the codes.

Much of the theoretical superiority of open prices in promoting

competition is vitiated by the fact that neither open prices nor competition operate as they should. These practical qualifications cannot be ignored.

Open prices do not always present a true picture. If producers wish to keep their terms secret, it is difficult to make them report, and once they do report it is not always certain that the figures are correct. If they send in a false price, the customers who know the truth would merely lose by exposing it. Even in the outstanding reporting systems there has been some of this "chiseling." There are more means of evasion than sheer falsehood. Producers may conveniently forget certain discounts, e.g., those resulting from advertising allowances to earn which little advertising was done, absorption of freight charges, "consigning" goods to a customer and paying storage until he uses them, over-liberal allowances on returned goods, sale of first grade goods as "seconds," etc. Successful price reporting requires, therefore, that all of these possibilities be exposed, and preferably that the various charges be standardized. In other words, price publicity may necessitate fixing many elements in the price structure (something which would not in fact eliminate competition, but would tend to concentrate its full force on the price level itself). Finally, even if producers are honest in their reports, problems enough remain—establishment of standards such that prices for the products of the various firms are comparable, speedy and proper handling of the mass of paper work, etc.[5]

No one would expect the competition between a heavy-weight and a fly-weight boxer to be beneficial, or economic competition to work out perfectly when the competitors differ greatly in size and power. Yet this is the actual business situation, and the chief defect of the open price system—that it may weaken competition and promote monopoly—arises from it. This criticism is pressed by two quite different economic groups—consumers and small producers—the former objecting to high prices, the latter to low prices.

The argument that price reporting promotes monopoly has been well stated from the consumer's point of view by his active champion, the Consumers' Advisory Board of the N.R.A. According to its report of February 19, 1934:

The basic difficulty as we see it is that open price systems, with or without waiting periods, identify the person or firm quoting the low price and thus facilitate the use of pressure to force his price up to the level generally desired in the industry.

The strength of this position must be conceded. Price fixing, whether it is based on an agreement among competing firms or on implied

[5] The filing agents of one of the first large industries to adopt an open price code accumulated hundreds of thousands of price reports within a few months.

threats by a powerful company against any competitor selling at less than the desired level, cannot be enforced unless the selling prices of the various firms are known. There has been a mass of evidence collected, in industries too numerous to mention, that pressure has been brought to bear on the price cutters. Even in the absence of manifestations of this kind, open prices tend definitely to discourage price cutting. If a company knows that price reductions will be published at once to all competitors, it will expect them to be met by all—but this takes away the incentive to make the cut in the first place. It is probable that the easier enforcement of price fixing and this natural discouragement to price cutting combine with the elimination of those reductions due to ignorance to make open prices tend in practice to be higher and more stable than secret prices—even though this cannot be proved by statistics. Insofar as these monopolistic factors deprive prices of their flexibility, the resulting price stability may be secured at the expense of more flexible factors in the economy.

One remedy proposed to prevent monopolistic tendencies is to conceal the identity of the seller, publishing only the price itself (perhaps only the average of the prices), along with the quantity if a past and closed transaction is involved. There is much to be said for this. The ticker service of the stock exchanges publishes only price and quantity, not naming the seller or buyer, and yet the facts given are quite sufficient for legitimate purposes, and the exchanges are recognized as free markets. The "bid and asked" quotations for stocks correspond to business price lists, with the firm quoting not named. But this type of quotation is satisfactory only because two conditions are fulfilled which are rare in ordinary trade. The sellers are all members of the stock exchange, approved and reliable; and the commodity dealt in is perfectly standardized and known. To announce that there has been a sale of oil burners at $200 is hardly as illuminating as to announce that there has been a sale of American Telephone common on the New York Stock Exchange at that figure. In the case of standardized commodities, like refined sugar or cement, the system could be employed, even though the absence of information on the identity of the seller would cause dissatisfaction. But in any industry in which products are not identical, the mere naming of the product "lets the cat out of the bag," and the big competitor could bring pressure to its heart's content. The rule, therefore, if adopted, would prevent makers of non-identical but closely competing commodities from learning each other's prices —or rather, since they would try to learn them by one means or another, it would prevent them from having reliable knowledge. Even in the case of standard products, the big company could usually discover

the price cutter under the secret price regime. "Non-identification" of sellers would simply reduce the frequency of this.

This issue requires a statement and analysis of the benefits expected from successful price reporting. (1) It prevents secret rebates and thus puts all buyers (of like quantities in like circumstances) on an equal basis, an effect which improves the process of competition by keeping sharp bargaining and unethical practices from being the keys to success. As long as price lists are identified to the filing agency, secret rebates can still be prevented by giving it enforcement powers. (2) It copes with the "lying buyer"—i.e., prevents the price cutting caused by a buyer falsely telling a seller that the latter's competitors had offered a product cheaper. Without direct word from his competitor, the seller cannot check the truth of this. Even with non-identification, it would be possible to discover whether the lower price was quoted, but, of course, not whether it was quoted by a given firm, or an important firm.[6] (3) It informs both the industries concerned and the general public of price trends, something with which non-identification would not interfere.

The complaint of the small producers against price reporting is that prices, instead of being drawn upward by the big companies, as feared by the consumers, are forced down to the lowest level. It is supposed that companies which find that they are being undersold will reduce their own prices to meet the competition. The result has been, in several industries, that the principal effect of open prices has been to reduce prices, even to the point of a price war. Price uniformity at a low level is especially injurious to small competitors. The large companies are likely to make a bigger appeal both to distributors and to the ultimate consumer by offering a full line of products, adequate service, assurance of future supply, and well-known and highly advertised brands. Complaints are made regularly that customers have switched their trade as a result of these factors. In the rubber footwear industry, for example, the fear of this occurring has caused three small companies to refuse to file their prices. When the Code Authority attempted to enforce the provision, they appealed to the Federal Trade Commission on the grounds that open prices, by causing this switch to the larger companies, constitute a "monopolistic practice," forbidden by Section

[6] So far as past transactions are concerned, the publication of the quantity would in most cases give a sufficient clue to the importance of the seller. The objection to non-identification in past transactions is small, just as the reason for it is less than in quoting current price lists. Business men, it may be remarked, usually prefer present reporting, but the filing of all invoices is almost a necessary part of an adequate reporting system in many industries.

3 (a) of the N.I.R.A. This suit was heard before the Commission in August and September, 1934, and placed on it a grave responsibility with regard to one of the chief price policies of the N.R.A.

This line of argument is based on the tendency of open prices toward uniformity which is indicated by both statistics and analysis. Buyers will hardly pay more to one seller when they can secure the article from another for less. This rule of uniformity, however, applies only to standardized products. If the product of a large company carries with it a guarantee, a wide selling appeal, or warehousing or engineering service, which that of the smaller company does not carry, the situation is not one of standardized products. The former company can command a higher price, and doubtless will, unless it definitely wishes to expand its market at the expense of the latter. Normally the company charging the higher price knew, even under secret prices, that it was being undersold—if it was content to let the matter alone then, why not now? If, however, price reporting first discloses this underselling to the big company, and it decides that it must meet price competition even though its own product is better, there is trouble in store for the small firm. Sometimes the result is healthy; the consumer gets a better product for the same price. Sometimes, however, the result cannot be called anything but "monopolistic."

Several remedies for the monopolistic tendencies which develop from the clash of open prices and the competitive structure of industry might be suggested. (1) The competitive structure might be changed by breaking up all industrial units into ones of equal size. (2) Price lists and sales can be reported without identification. Considered by itself, this might be a wise plan—it would make possible most of the benefits to be expected from open prices, while avoiding the greatest dangers. It would, however, work injustice as between various industries (e.g., standardized products vs. non-standardized products), and, above all, would be a backward step. Secrecy is slow in losing its grip on industry, and knowledge and publicity are slow in coming in; but the trend is clear. The argument that "prices must be kept secret in order to make price fixing more difficult" would be hard to defend. (3) Open prices can be kept, and an effort made to abolish as many of their abuses as possible (while realizing that some will remain in any case). Firearms are not abolished because they can be used against human beings, and, to come nearer home, standardization of products by trade associations is not condemned because price fixing is easier when the product is standardized.

Many of the abuses of open prices are well recognized. Suggestions by the reporting agency as to what prices to file, association of price reporting with the fixing of terms of sale, concealment of price lists

from any but members of the industry—these and many more are definite aspects of open prices which work toward monopoly and which can be eliminated. Business men already realize, to judge by the poll taken among the Code Authority members who attended the "field day" in March, 1934, that price lists should go to customers too; and they can easily be made to realize that all forms of price fixing not definitely permitted by approved codes are still under the ban of the anti-trust laws. A few words should be said on the two abuses which have caused by far the most trouble under the N.R.A.—the waiting period and government bids.

There are strong argument for a waiting period between the filing of prices and their effective date. Sellers are helped, since (1) the reporting agency has time to get the list out to the entire membership; and (2) "surprise moves" intended to secure special contracts, which always tend to demoralize the market, are eliminated. In other words, only by a waiting period are truly "open" prices assured in the sense that each competitor always knows what the rest are charging. Buyers too (1) get time to make complaints and adjust their plans—e.g., buy in advance—before the new schedules go into effect, and (2) are protected against underselling by competitors able to get raw materials without notice at lower prices. According to opponents of the waiting period, (1) it enables the dominating companies to bring pressure to bear on the smaller ones to withdraw price reductions or come into line on price advances; (2) it gravely hampers firms which have to make price changes quickly—e.g., to secure cash at once; and (3) it "freezes" prices for a given number of days. The economic argument that it ties up individual companies in their price policies, which should be flexible, is controlling. N.R.A. Office Memorandum 228 has condemned the waiting period except where the industry can show special circumstances to justify it.[7]

One of the most extreme forms of price fixing with the aid of open prices has been the prior exchange of bids destined to be submitted, sealed, to the Federal Government, States, cities, or other public bodies. The result has been to encourage identical bids. There is nothing in the open price scheme which would permit sellers to "gang up" on a named buyer and tell each other beforehand what they will charge him on his order. If he is to be charged the same prices as every one

[7] Although waiting periods are not to be compulsory, individual companies may still announce their prices effective at a later date. Office Memorandum 228 suggests that price lists "become effective immediately upon receipt thereof by said agent," thus establishing a waiting period of at least the length of time necessary to notify the agency, but not preventing a company from announcing a new price list to the trade and filing it with the agency much later.

else (in his class), he has no complaint, but in that case there would be no need to exchange the bids specially. The weight of evidence seems to be against the prior exchange of bids; but there is enough doubt surrounding the whole subject that a careful investigation of the problem of public purchases is being made by the Research and Planning Division of the N.R.A.

The subject of open prices has drawn relatively little attention from economists and statisticians. Few books have even referred to it, outside of the ten or twelve on the subject of trade associations, and some of these have discussed only the legal angles. Out of twenty economists circularized by the N.R.A. early in 1934, nearly all urged study of the subject, but none took a definite stand (suspicion of price fixing under the guise of open prices was expressed by eight). It seems fairly clear that the most fruitful type of inquiry is not that of general statistics, but of case studies. Some of these are in progress at the Recovery Administration. It can be said, however, that there is room for valuable work by economists and statisticians in this field.

III. OPEN PRICES UNDER THE NATIONAL RECOVERY ADMINISTRATION

On June 16, 1933, the United States Congress passed the National Industrial Recovery Act, which, according to Senator Robert Wagner,[8] was, among other things, to legalize current price reporting. Industry was very willing to take advantage of this purpose of the law. The fourth code of fair competition under this law to be approved by the President (on August 4, 1933), that for the electrical manufacturing industry, was the first to include an open price provision. Article x provided that makers of standard products

file with the supervisory agency a net price list or a price list and discount sheet, as the case may be, individually prepared by him showing his current prices, or prices and discounts, and terms of payment, and the supervisory agency shall immediately send copies thereof to all known manufacturers of such specified product.

The Iron and Steel Code, approved on August 19, was the next to contain this type of clause. In both of these giant industries competitors had in the past possessed a working knowledge of each other's prices, but in neither had there been in recent years a definite and reliable exchange of prices—and especially of discounts—through an association. Many other industries followed the example of these two when they came to propose their codes to the National Recovery Administration. On August 8, 1934, Code No. 500 was approved, and within the same year's time 143 supplementary codes, covering sub-

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[8] See Appendix I.

groups within major industries, had been accepted. Of these 643 major and minor industries, 383, or 60 per cent., had adopted price reporting (293 in codes, and 90 in supplements).

The open price clauses in the codes may be classified according to whether they require reporting of prices on past and closed transactions or reporting of current quotations. There are 20 in the former group and 363 in the latter (including some which provide for past reporting also). The filing of prices on actual sales has aroused little opposition except when parties to individual transactions are identified to the competitors of the seller. The real controversy under the N.R.A. has raged over the exchange of *quotations*, and it is this aspect chiefly which the present paper will consider.

One variation of the exchange of quotations is introduced by the so-called "waiting period." Codes with this type of provision state that price changes are to be filed with the clearing-house a given number of days in advance of their effective date. Two hundred and seventy-eight codes, 77 per cent of those under which quotations are filed, include such a waiting period. The length varies:

Number of days in waiting period	Number of codes
30	1
20	4
15	5
10	141
7	12
6	2
5	89
3	12
2	1
1	6
flexible	5
	278

Twenty-eight of these waiting periods apply to price declines only.

The usual style of price reporting is that used in the Electrical Manufacturing Code—the filing of prices with a central agency and their redistribution to the other members of the industry by the agency. In some instances, however, variations are found. Occasionally the requirement is simply that the members must "publish" or "post" their quotations; occasionally the quotations, although sent in to the filing agency, are kept there without being redistributed. These exceptions leave 286 instances in which it is definitely stipulated that price lists are to be mailed out to all members. In only 135 codes is any provision made for giving customers access to the price lists. The Code Authority, or group of employers (usually including a government representative) administering the code, is made the clearing-

house for exchange of quotations in 314 cases, the trade association of the industry in 34, and an impartial agency (such as a credit bureau or accounting firm) in 15.

A classification of codes by type of industry indicates that price reporting is more popular among these selling raw materials and producers' goods than among those serving the ultimate consumer:

	OPEN PRICE CODES				CODES WITHOUT PRICE REPORTING
	With waiting period	Without waiting period	Past price reporting	Total	
Raw materials	19	7	1	27	2
Fabricated products sold to producers	198	41	14	253	129
Total producers' goods	217	48	15	280	131
Fabricated products sold to consumers	45	24	4	73	73
Services	16	13	1	30	56
Total consumers' goods and services	61	37	5	103	129
Total codes	278	85	20	383	260

It would be naïve to assume that the zeal of organized industry for open prices is based solely on a belief in the values of publicity. The business man who favors price reporting does so normally because he thinks that it will tend to stabilize prices. Occasionally, as in the twenty-eight codes whose waiting periods apply only to declines, the desire to hinder price cutting without blocking increases becomes plain. Often the open price clause in the code is definitely tied in with other rules which even more clearly work in this direction. For example, 257 of the 363 codes providing for reporting of quotations forbid sales of the product below cost of production (individual cost of the seller in nearly all cases, but occasionally cost of a representative firm or some other uniform level). Members of these industries hope that the publication of prices will enable the Code Authority to know which companies are violating the rule against selling below cost, and they commonly view this as the principal purpose of price reporting. Again, open prices are related in many cases to the classification of customers. Members of the industry are expected to announce the prices applicable to each group of buyers—jobbers, retailers, etc.,—and to give all members of each group the same prices. Extra discounts to large purchasers or cooperatives can thus be eliminated—an aim close to the heart of many sellers. In only thirty-one of the open price codes

is a definite classification set up which members are obligated to follow; but in many more it is assumed that each member will classify his own customers when he publishes his prices, and the Code Authority sometimes distributes price filing forms with all types of buyers classified. These provisions—open prices, prohibition of sales below cost, fixing of resale prices, fixing of discounts, etc.,—can easily be combined to establish a reasonably effective price fixing scheme. This explains without doubt a good part of the enthusiasm which they have excited among industrialists.

It is only natural that the zeal of organized business for open prices, based on a belief in their efficacy in preventing price cutting, should be matched by an equal zeal against them, based on the same belief, on the part of representatives of the consumers. In both groups, to be sure, there has been a dissenting minority. Many business men still prefer the advantages of keeping their own price schedules secret to those of knowing the schedules of their competitors; while many spokesmen for the public or consumer interest have declared for open prices. Nevertheless, the conflict remains, and the progress of price reporting under the N.R.A. has met a continual fire of criticism. This criticism first came to a head on January 9 and 10, 1934, when a "field day" was held at Washington for complainants against the price increases which had come since the passage of the N.I.R.A. At this hearing complaints were presented against forty codes, thirty of which provided for open prices, but in only seven instances were open prices actually in effect at the time of the price situation or movement against which complaint was made. Although no case against open prices was established, the N.R.A. was led by the widespread dissatisfaction to review the whole question carefully. An Office Order was issued on January 27, 1934, to the following effect:

As a result of the price change hearing a study is being made of open price associations. This study particularly involves the waiting period before filed prices become effective. Therefore, any provision for a waiting period in codes not yet approved will be stayed in the Executive or Administrator's Order of Approval for sixty days, or pending completion of the study.

Since the study launched at that time resulted in a condemnation of the waiting period, this "stay" was never lifted. The only waiting periods in effect today (October, 1934) are the ninety-eight which had been approved on or before January 27, 1934. Under these codes only is it a violation for members to file price revisions effective at once or within less than a given number of days. Meanwhile, during the progress of the study mentioned, two more "field days" were held at Washington to discuss the N.R.A. At both the price problem most hotly debated was that of price reporting. Sixteen speakers opposed it

and only six favored it at the hearing for the general public, which
lasted from February 27 to March 2. When the Code Authority
members convened, however, from March 5 through March 8, the
point of view of the sellers had its innings—sixteen speeches favoring
open prices and only three opposing them (with five indecisive). Be-
tween these hearings, on March 4, the Consumers' Advisory Board of
the N.R.A. had presented a preliminary report adverse to open prices,
especially when quotations were published in such a way as to identify
the seller. In April and May the investigations being conducted by the
Research and Planning Division and the Consumers' Advisory Board
were brought to completion, and on June 7 the new policy of the
N.R.A. was announced, under the title "Office Memorandum 228."

The spirit of the Memorandum was to encourage open prices as an
aid to free competition and to prevent their use for the price fixing
feared by the Consumers' Advisory Board and complained against at
the "field days." It suggested that industries desiring open price clauses
in their codes should include the following points: filing of prices to
be with a "confidential and disinterested agent" rather than with the
Code Authority, immediate distribution of prices filed to all members
of the industry and to all customers willing to share the cost, and elimi-
nation of waiting periods—except that a price once reduced was not to
be raised again for forty-eight hours. A warning was included against
advance agreements on prices to be filed, a principle which had been
violated by a number of industries under the mistaken impression
that the anti-trust laws had been completely repealed by the N.I.R.A.,
and that price fixing was now legal even without sanction of the in-
dustry's code. The purpose of the "disinterested agent" clause was to
prevent a clique within the industry, acting as the Code Authority,
from obtaining advance or undue information from the price filings of
the others or from effecting price fixing agreements. The higher type
of Code Authority and trade association secretary does himself act
impartially. The purpose of the forty-eight hour clause was to prevent
a producer from gaining an unfair advantage, and giving a customer
an unfair advantage, by reducing his filed price just in order to make
one sale at a heavy discount, and then raising it again.

The official adoption of Office Memorandum 228 did not mean the
immediate suspension of price clauses in conflict with its principles.
It was intended to apply these principles to codes not yet approved,
but to amend already approved codes only gradually and by agree-
ment with the industries concerned. Naturally very few of the latter
have actually been revised (up to October, 1934), the Memorandum
not being to the liking of many industries. It was not until August
that its principles began to find their way into the new codes, since

the Administrator took the position that codes which had reached their final stages when Office Memorandum 228 was announced should not be subjected to drastic amendments without the industry's consent. Consistency in the open price policies of the codes of fair competition is, therefore, not yet established.

IV. EFFECTS OF OPEN PRICES ON PRICE INCREASE AND PRICE STABILITY

The statistical work which has been done to determine whether open prices fulfill the hopes of their supporters and the fears of their opponents by being higher, more stable, and more uniform (as between different sellers), than secret prices will be reviewed at this point. First let it be said that the word "secret" is applied to all prices which are not regularly exchanged between the sellers, even though many of these prices actually become as well known as the "open" ones— through trade journals, easy access to price lists, reports from buyers, etc. The issue under discussion has not been whether these types of publicity should be permitted, but whether associations would benefit by handling the exchange of prices and whether the law should permit this activity, or even, through codes of fair competition, compel the members to cooperate. The proper comparison, therefore, is between the prices thus exchanged and all prices not thus exchanged.

Professor Milton N. Nelson made the first brief study of the effect of price reporting on prices,[9] drawing his data chiefly from the court record in the suit against the American Hardwood Manufacturers' Association.[10]

He found that prices obtained by members of the association did not average higher than those obtained by the one non-member whose quotations could be secured, and that the prices obtained by three leading members were not uniform when compared week by week. He admitted that the force of his conclusions (which were, however, the same drawn by market observers at the time) was weakened by the narrow scope of his data and by the dominance of the market in the year in question, 1919, by runaway demand.

A much more extensive investigation was made by the Federal Trade Commission, under instructions from the United States Senate to inquire

to what extent, if any, the effect of such open-price associations has been to maintain among members thereof uniform prices to wholesalers or retailers, or to secure uniform or approximately uniform increases in such prices.[11]

[9] "Open Price Associations," *University of Illinois Studies in the Social Sciences,* Vx, No. 2 (June, 1922), pp. 176–182.

[10] *American Column and Lumber Co. et al. v. United States,* 257 U.S. 377, (1921).

Since the available data did not make possible the study of price uniformity or increase suggested, the Commission's staff devoted its attention to price stability.[12] This work is doubly worth discussion since it has remained buried, in obscure form, in a bulky government document.[13]

The raw material consisted of the wholesale prices collected monthly by the United States Bureau of Labor Statistics. There were available for the complete period used, 1921 through 1925, 272 non-agricultural series. Forty of these are stated to have come under the influence of open price associations at some time during these five years. Each commodity was given three indices of price variability: (1) a crude index—the average monthly change for the fifty-nine months beginning in February, 1921, divided by the average price for the five years; (2) an index adjusted for secular trend—obtained by deducting from the crude index 1/59 of the net change between January, 1921, and December, 1925; and (3) an index adjusted for seasonal variation, which may be disregarded, since little use was made of it. The mean of the indices for the 40 open price commodities, adjusted for trend, was .0255—i.e., the average monthly movement of each commodity was 2.55 per cent of its average price.[14] The mean for the 232 secret price commodities was .0262, showing slightly less stability. Each of the commodities was then assigned an "ordinal," to indicate its rank in order of increasing stability. The mean open price ordinal was 131.2, and the mean secret price ordinal 147.2, the latter being apparently more stable. Evidently the index and ordinal yielded opposite results. In the words of the report,

The meaning of this apparent mathematical inconsistency between the two comparisons is that there is no significant difference in degree of stability between the two groups as represented in these figures.[15]

In a second study, the Commission added to the original forty open price commodities twenty-four more, data on sixteen of which had only just become available, while series for the remaining eight covered only two to four years instead of the full five. The mean index of variability

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[11] "Open-Price Trade Associations," *op. cit.*, p. 4. The statistical study is found on pp. 93–121 and in Appendices C to F.

[12] *Ibid.*, p. xix.

[13] It is mentioned in Henry R. Seager and Charles A. Gulick, Jr., *Trust and Corporation Problems*, New York: Harper and Brothers, 1929, p. 320.

[14] "Open-Price Trade Associations," *op. cit.*, p. 98. The results of the study are tabulated chiefly on pp. 98 and 100.

[15] *Ibid.*, p. 98. Significance was not tested quantitatively by the Commission in any case. The ordinals given here are slightly in error, since they were taken over from the ranking of 296 commodities (see next paragraph). They should be 123.4 and 138.8.

for the 64 commodities was .0207,[16] indicating greater stability than the .0262 index of the 232 secret prices. In this case the result was supported by the ordinals—153.2 and 147.2, respectively. A new measure, however, introduced at this point, that of "zero changes", contradicted this conclusion. The open prices, on the average, showed no change in 41.6 per cent of the months, while the secret prices showed no change in 47.9 per cent. None of these differences are mathematically significant.[17]

The zero changes measure is the most logical, since it isolates frequency of price changes, instead of mixing in the element of their size.[18] Size should not usually be considered, since the deductions to be drawn from it are not certain. A moderate degree of stability achieved by an association would mean small changes, but a greater degree would mean large ones, since changes too long delayed would have to be considerable to put the price back in line with fluctuating supply or demand.

In addition to its general studies, the report contained five tables which developed the contrast in stability between open and secret prices within different commodity groups, according to all three measures.[19] To a reader of these tables it appears that degree of stability is influenced more by the nature of the product than by open price activity, and that it increases as the product approaches the ultimate consumer.

A careful analysis of the commodities chosen for the open price grouping suggests the desirability of some revision. Four of the open price associations did not actually exchange prices during the five-year test period.[20] A transfer of their commodities to the secret price group reduces the number of open price series from 64 to 38:

	Open	Secret	Probability of chance result
Index of variability			
22 open vs. 250 secret	.0322	.0256	.2 to .3
38 open vs. 258 secret	.0251	.0250	.9
Zero changes index			
38 open vs. 258 secret	32.3%	49.0%	.01
31 open vs. 258 secret	40.0%	49.0%	.2 to .3

[16] Had the eight commodities whose time periods were not exactly comparable been omitted, a calculation shows that this figure would be .0222.

[17] The probability of the difference resulting from chance is .1 to .2 in the case of the index and .3 in the case of the zero changes, using the table in R. A. Fisher, *Statistical Methods for Research Workers*, Edinburgh and London: Oliver and Boyd, 1930), p. 139.

The third comparison indicates that open prices are less stable, but the fourth shows the situation if seven brands of flour are omitted. The greater instability according to the zero change index is largely due to the concentration of the open prices used by the Commission in a few volatile groups—flour (7 items), cotton yarn (4), and lumber (13).

In addition to its price studies, the Commission was able to find some statistics bearing on the relative stability of output in open price industries and industries with statistical, but without open price, associations:[21]

	Open price associations	Associations not reporting prices	Probability of chance result
Production			
Number of series	19	16	
Mean ordinal	18.6	17.3	
Mean index	.227	.204	.5 to .6
Shipments			
Number of series	27	21	
Mean ordinal	20.5	29.6	
Mean index	.202	.287	.02
Sales			
Number of series	34	17	
Mean ordinal	23.4	31.2	
Mean index	.232	.298	.05 to .1
Stocks on hand			
Number of series	18	15	
Mean ordinal	14.8	19.6	
Mean index	.138	.206	.1 to .2
Unfilled orders			
Number of series	12	15	
Mean ordinal	13.8	14.2	
Mean index	.246	.474	.01

In this study stability is indicated by a lower ordinal and a lower index, and in each of the five cases the two measures agree. Production

[18] In fact, there is a fair degree of (inverse) correlation between the index of variability and the zero changes index. The coefficient of correlation is $-.73$ (probable error .018).

[19] "Open-Trade Associations," op. cit., pp. 96, 99, 101, 102.

[20] These associations were the Copper Institute, Corn Derivatives Institute, Industrial Alcohol Institute, and Wool Institute. See Ibid., pp. 118–119, 465–467.

[21] Derived from Ibid., pp. 116–117, by combining "intensive open price" with "other open price." The index for each series is the mean of the differences between the logarithms of the high and low figures for each year. All series are in quantities rather than values.

appears to be slightly less stable in the open price industries, but the result is as likely as not to be due to chance. The four other series show the open price associations to have had a stabilizing effect, and in two cases the results may be taken as significant. It might be thought that open prices, being more stable, encourage customers to order evenly, were it not that the middle link, stability of open prices, remains unproved. The seasonal element, for which no correction was made by the Commission's analysts, may account for some of the differences. In the opinion of the Commission, an opinion which might be applied also to the price studies:

The inexactness of the data, together with the closeness of the differences, permits only the conclusion that open-price associations produce no very distinctive results for their industries.[22]

A few case studies of specific open price associations in the period before the N.R.A. have been made by the present writer. One organization for which usable figures are available is the Sugar Institute, whose prosecution by the Department of Justice at a time when it was a model for other trade associations did much to stimulate industry to the insistent demand for relief from the anti-trust laws, which was one factor behind the Recovery Act. The following table presents figures for the five years before and after the establishment of the Sugar Institute as an open price association (at the beginning of 1928):[23]

	Price uniformity		Price stability		Stability of operations
	Average number of simultaneous quotations, ten refineries	Average range of quotations, in cents per 100 pounds	Number of price changes during year	Average size of price change, in cents per 100 pounds	Mean percentage deviation of monthly refinery activity from annual mean
1923	2.5	28.4	67	20.0	
1924	3.1	27.4	48	16.4	
1925	2.7	18.5	43	9.4	18.9%
1926	2.9	21.0	42	10.1	12.4
1927	2.7	18.6	36	11.1	16.6
1928	1.4	5.0	28	9.8	13.5
1929	1.1	10.0	19	16.1	15.5
1930	1.3	22.7	22	13.4	15.7
1931	1.0	7.5	21	10.0	19.6
1932	1.1	5.6	16	9.4	16.2

[22] *Ibid.*, p. 117.
[23] Derived from figures appearing in Willett & Gray's *Weekly Statistical Sugar Trade Journal.*

The only trends are the decreasing number of quotations being made at one time, the decreasing range between the highest and lowest quotation, and the decreasing number of price changes made during each year. Although they point to increased uniformity and stability, they are not absolutely conclusive. The variety of quotations before the Institute was often more apparent than real, sales seldom being made at any quotation but the lowest. Thus the establishment of open prices merely meant that quotations were made in a more orderly and reliable manner, as a result of which the latent price uniformity (which always exists in the case of a standardized commodity when buyers and sellers all know the market condition and are all in a position to act) was brought to the surface.[24] Similarly, the increasing stability may have been due to other aspects of the Institute's work (and partly to a declining range in the prices of raw sugar) rather than to price reporting.

The investigation made in the spring of 1934 by the Consumers' Advisory Board of the N.R.A. developed, among other very interesting results, some data bearing on the influence of open prices on uniformity. A comparison of the first lists filed after adoption of the code with the latest lists available (usually those for February, 1934), for as many industries as answered a questionnaire and supplied comparable figures, gave results which may be expressed as in table on p. 59.[25]

It appears from these figures that uniformity tends to result from price reporting, that it tends to increase after filing of the first lists, and that prices generally tend to cluster at the top with a few companies underselling rather than at the bottom with a few companies charging more or in the middle with some charging more and some less. This third point indicates a general harmony, with a fringe of price cutters.

Another branch of the N.R.A., the Price Unit of the Research and Planning Division, has made a number of statistical studies of the effect of the open price codes. One group of these studies will be reported here.

A comparison was made of the relative increase of open and secret prices during the first year of the N.R.A. on the basis of the monthly wholesale prices compiled by the United States Bureau of Labor Statistics. In order to avoid comparing products of quite different eco-

[24] Real uniformity was increased in one respect not disclosed in the table. Secret discounts were almost entirely eliminated, so that the existence of a single selling price in the market, which before 1928 might have meant anything or nothing, came to mean that there was only one actual price level.

[25] The number of products covered by the reports of a single industry ranged from 3 to 537. Fractions mean that the industries in question did not show a "plurality" for any single degree of identity, but a "tie" for two.

	Number of products	Number of industries having plurality of products in class
First list filed		
All prices identical	1321	23
More than half identical	742	7
Half or less identical	55	1
None identical	211	7
Total comparable data	2329	38
Latest list filed		
All prices identical	1985	28
More than half identical	189	3½
Half or less identical	31	2
None identical	172	4½
Total comparable data	2377	38
Both lists, identical prices only		
Identity found at highest price	638	8
Identity found between highest and lowest	219	7½
Identity found at lowest price	234	8½
Total data included	1091	24

nomic character, none were included except those in "sub-groups" of the Bureau's index containing at least two open and two secret price series. There remained fourteen (out of forty-five) sub-groups, including 181 products under open price codes approved before the end of July, 1934, 130 products under other approved codes, and 52 not yet under codes at that date. The first two open price codes came into force on August 15 and August 29, 1933, but price lists were not filed at once. August, 1933, therefore, is the last "pre-code" month. The frequency distribution of the movements of the full 363 price series between significant dates as in table on p. 60.

A number of means were computed (but not weighted, since there is no reason to believe that the influence of price reporting is correlated with the importance of the commodities affected) as in the table on p. 60.

The conclusion from both tables is that commodities later to be covered by open price codes declined more slowly than other commodities down to February, 1933 (the low point of the whole index), recovered more slowly up to August, 1933, but commenced to gain ground thereafter. Among the open price codes these tendencies are most evident in those with waiting periods and least evident in those

NUMBER OF PRICES CHANGING AS INDICATED

Percentage change	Decline, 1926 (average) to February, 1933		Advance, February, 1933, to August, 1933		Advance, August, 1933, to August, 1934	
	Open	Secret	Open	Secret	Open	Secret
−50 to −40.1						3
−40 to −30.1					1	2
−30 to −20.1			1		5	6
−20 to −10.1	2	1	2	3	11	30
−10 to −.1	4	4	10	7	33	26
0 to 9.9	18	9	86	59	75	60
10 to 19.9	29	21	31	20	36	32
20 to 29.9	34	26	14	17	15	10
30 to 39.9	35	23	15	13	2	7
40 to 49.9	27	30	7	12	2	2
50 to 59.9	20	33	3	13		3
60 to 69.9	7	23		12		
70 to 79.9	5	10	2	7	1	1
80 to 89.9		2	1	4		
90 to 99.9			1	5		
100 to 109.9				5		
110 to 119.9			1	1		
120 to 129.9			1	2		
130 to 139.9			1	2		
140 to 149.9			4			
150 to 159.9						
160 to 169.9						
170 to 179.9			1			
Total prices	181	182	181	182	181	182
Median	31.0%	42.3%	9.0%	21.2%	5.4%	4.0%

MEAN PERCENTAGE MOVEMENT

	Number of prices	Decline, 1926 (average) to February, 1933	Advance, February, 1933 to August, 1933	Advance, August, 1933 to August, 1934
Codes with waiting period	73	25.7%	8.5%	7.7%
Codes without waiting period	84	33.1	16.7	5.4
Codes with past price reporting	24	40.2	55.7	−5.1
All open price codes	181	31.1	18.5	4.9
Other codes	130	39.8	35.2	.5
Prices not under approved codes	52	40.0	23.3	8.2
All non-agricultural prices	707	36.3	13.0	8.1

with past reporting only (these last being dominated by semi-manu-
factured goods, a group whose recovery came earlier than that of the
rest). Not all of the contrasts shown are significant, but in the key
case—the difference between the 4.9 per cent increase of open prices
and the .5 per cent increase of secret prices after adoption of the codes
began—the probability of a chance result is only .01. The relatively
rapid increase (8.2 per cent) of the commodities not under approved
codes is due to their concentration among certain food products whose
rise has continued during 1934. The most reasonable explanation of
the whole movement is that the open price codes are found among the
industrial commodities—iron and steel, electrical equipment, industrial
machinery, building materials—whose prices are more sluggish than
those of other comodities. If this is so, the conclusion is that the
whole price increase has slowed down appreciably, but that these slug-
gish items are still catching up. The relatively slow-moving nature of
these commodities is probably due partly to their position on the

MEAN PERCENTAGE MOVEMENT

	Open	Secret	Decline, 1926 (average) to February, 1933		Advance, February, 1933, to August, 1933		Advance, August, 1933, to August, 1934	
			Open	Secret	Open	Secret	Open	Secret
			%	%	%	%	%	%
Fruits and vegetables	13	6	31.2	46.8	8.6	43.5	10.4	20.2
Other foods	14	22	40.2	49.5	15.3	23.4	5.8	9.9
Clothing	2	18	38.0	39.3	16.7	36.9	16.0	3.6
Cotton goods	8	24	43.5	48.8	127.8	79.1	−11.4	−1.9
Knit goods	3	5	26.0	55.4	13.3	52.8	−.8	−6.3
Silk and rayon	3	10	69.2	69.0	43.8	27.4	−28.0	−16.0
Other textile products	14	2	45.1	38.9	32.6	26.1	−2.9	−14.6
Iron and steel	46	17	21.5	16.2	6.1	8.8	8.5	5.1
Nonferrous metals	9	11	46.4	43.6	39.3	40.0	2.1	5.2
Paint and paint materials	7	24	41.1	28.5	18.1	11.9	14.4	4.0
Other building materials	18	6	26.5	26.7	11.5	3.9	3.9	−4.8
Furnishings	19	20	26.0	32.4	7.5	22.4	7.2	.9
Paper and pulp	7	4	37.6	37.5	19.1	40.0	6.7	−5.5
Other miscellaneous	18	13	25.1	43.2	5.4	26.6	4.2	5.9
Mean of sub-group means			37.0	41.1	26.1	31.6	2.6	.5

road from producer to consumer (most of them being highly fabricated, although technically "producers' goods") and partly to some form of control exercised by the members of the industries concerned. The known existence of market control in many open price industries strengthens the belief that open prices tend to be desired by an industry in order to assist in such control.

A classification of price movements according to commodity subgroups weakens conclusions drawn from the more rapid rise of open prices under the codes, since secret prices rose further or declined less in seven of the fourteen sub-groups as in table on p. 61.

Although the codes have been in effect too short a time to allow a good study of price stability under them, some efforts have been made by the Research and Planning Division to analyze this factor. The next table shows the relative number of price changes under open and secret price codes between the dates of their adoption and August, 1934:

Percentage of months in which a price change occurred	Number of prices		Percentage of total	
	Open	Secret	Open	Secret
0%	51	32	28.2%	24.6%
1.9 to 9.9	4	3	2.3	2.3
10 to 19.9	20	20	11.0	15.4
20 to 29.9	20	15	11.0	11.5
30 to 39.9	19	13	10.5	10.0
40 to 49.9	14	8	7.7	6.2
50 to 59.9	12	7	6.6	5.4
60 to 69.9	6	9	3.3	6.9
70 to 79.9	3	2	1.7	1.5
80 to 89.9	4	4	2.2	3.1
90 to 99.9	2	3	1.1	2.3
100	26	14	14.4	10.8
Total	181	130	100.0	100.0

There is no significant difference between the stabilities of the two groups.[26]

A few of the obstacles met in these studies of the codes may be mentioned, with the remark that they are typical of those met in all

[26] The mean percentage of months in which changes occurred was 36.2 per cent for the open prices and 35.8 per cent for the secret prices. The respective means for the (varying) periods between February, 1933, and the effective dates of the codes were 43.6 per cent and 49.3 per cent. This difference is not significant, and the increase in stability between pre- and post-code periods is due to general flattening out of the rate of price advance.

investigations of open prices. (1) It is impossible to secure exact classes
of open and secret prices for comparison. The dates on which price
reporting began and ceased are not definitely known, nor are the vari-
ous degrees of openness and secrecy in the secret price industries. (2)
It is impossible to shut off the influence of other factors which ac-
company open prices—the various price movements due to monetary
policy, business recovery or recession, price controls, etc. In particular,
open prices are affected by other parts of codes as well as by the price
reporting provisions. (3) The price data are unsatisfactory; e.g.,
indices measure price lists and ignore discounts, and yet price move-
ments have often taken the form of changes in discounts only; the
number of prices which can be attributed to specific codes and for
which series are available is limited; etc.

Then there is the question of factors other than openness of price,
code provisions, and general administrative policy of the Govern-
ment, which enters into the determination of price stability. Studies
made by the Research and Planning Division indicate that there are
at least four major factors which influence price stability, namely:

1. Rate of variation in the cost of production. This variation in
cost of production usually depends upon varying costs of raw materials,
and upon varying costs of labor. Examples are raw and semi-finished
materials of a standardized nature such as copper, pig iron, cotton
yarn, brass products, etc.

2. The number of independent buyers and sellers handling the bulk
of the commodity, that is, the number of effective buyers and sellers.
For a relatively small number of sellers the variation in price tends
to be proportional to the number of independent sellers, thus increas-
ing as the number of independent sellers increases. The number of in-
dependent buyers is also an important factor in determining price
variation. In fact, the rate of variation of price tends to be proportional
to the reciprocal of the number of buyers. To summarize, the varia-
tion in price depends to considerable extent upon the ratio of the
number of sellers to the number of buyers.

Products and services rendered according to specifications by bid-
ding such as pressed metal, drop forging, job printing, construction,
etc. are essentially open price products. Here there is one buyer and
many sellers. On the other hand for consumer necessities of a highly
standardized nature such as soaps, matches, etc., there are many
more buyers than sellers. Manufactured products of necessity char-
acter include certain steel products, brass products, electrical goods, etc.

Markets in which there are very few sellers are usually character-
ized by nominal prices which are extremely stable, and which are
shaded by these sellers in order to get large orders. What movement

does occur in the prices in these markets is probably largely the result of suspicion on the part of certain sellers that others are allowing sharply increased discounts to customers. On the other hand, in an industry where there are a great many sellers, and prices fluctuate widely, prices are almost always quoted in open markets when the product is standardized enough to justify this, and these prices are not subject to substantial secret discounts, but amount pretty closely to actual minimum purchasing prices. Consequently an open price association or price listing agreements could possibly tend in the future to stabilize even more those prices which have been comparatively stable in the past, without giving any relief from price decline to those industries that have had highly variable prices in the past. It appears, therefore, that open price and price listing agreements as written in some of the codes may well tend to aggravate rather than improve the price inequalities which have produced so many economic pains. Nevertheless, the open price system has some obvious advantages. It leads to competition above-board and prevents insidious price slashing and the development of obnoxious trade practices which very quickly spread throughout the industry and may, when competition is very keen, lead to a collapse of prices.

Selling or buying collusions have the same effect upon price variations as reduction in the number of effective buyers or sellers. Collusions or monopolies may occur in many classes of commodities, for example, complex or specialized products, machinery, etc., protected by patent rights, franchises, etc.

3. Existence of a significant export market greatly increases domestic price stability. Existence of a significant import market greatly decreases price stability. Export commodities of the United States include, for example, electrical goods, aluminum, etc. Import commodities include, for example, raw materials such as rubber, manganese, etc., and finished goods such as shoes, clothing, etc.

4. Degree of competition between substitute commodities or services. This includes either luxuries competing with more staple products for the consumer's dollar or products competing with functional substitutes from some other industries. Lumber competes with brick, cement, steel and stone. Rayon competes with cotton and silk. Services include cleaning and dyeing, barbering, transportation, etc.

It is impossible to deny that the statistical work in this field has been disappointing. At least one study points in each direction as regards the four disputed points—price uniformity, stability and increase, and production stability—and yet none of them rests on a really strong foundation. The only point that appears to be backed by good evidence is the increasing uniformity of prices under price reporting.

APPENDIX I
LEGAL STATUS OF PRICE REPORTING

The open price activity of trade associations, like much of their other work, always ran the danger of conflict with the anti-trust laws, especially the Sherman Anti-Trust Act of 1890. The ruling principle of this body of law is that contracts which are in unreasonable restraint of trade are illegal. This principle was used by the United States Supreme Court in 1921 and 1923 to condemn the price reporting and trade statistical work of associations in the hardwood lumber and linseed oil industries, but was not held in 1925 to cover the similar plans of the cement and maple flooring associations.[27] An analysis of these decisions indicates that they placed the seal of legality on the reporting of prices in past and closed transactions, unless the reports identified buyer and seller in each case without some evident and legitimate reason. The reporting of current price quotations, however, while not definitely declared illegal by the Court, was treated as one piece of evidence that a price fixing agreement might exist. Price fixing agreements themselves have long been held to be contracts in unreasonable restraint of trade.

The legal status of price reporting is believed to have been very much improved by the passage of the N.I.R.A. in June, 1933, although there are no court decisions interpreting the Act on this point. The principal purpose of this statute was to increase employment and wages, but a subsidiary purpose was the elimination of unfair practices in competitive business. Since trade associations had been working toward this goal for many years, the Act lent the endorsement of Congress to this part of their program. With regard to open price work in particular, the chief sponsor of the law on the floor of the Senate specifically mentioned the legalizing of current price reporting as one object its framers had in mind.[28]

National Recovery Administration Washington

[27] *American Column and Lumber Co. et al. v. United States*, 257 U.S. 377; *United States v. American Linseed Oil Co. et al.*, 262 U.S. 371; *Maple Flooring Manufacturers' Association et al. v. United States*, 268 U.S. 563; *Cement Manufacturers' Protective Association et al. v. United States*, 268 U.S. 588.

[28] Senator Robert F. Wagner, *Congressional Record*, June 13, 1933, p. 5939.

DEMAND FUNCTIONS WITH LIMITED BUDGETS

By Harold Hotelling

I

IN THE study of related commodities it is important to consider the properties that may reasonably be attributed to the demand functions. Taking these functions in the form

$$(1.1) \qquad p_i = f_i(q_1, q_2, \cdots, q_n), \qquad (i = 1, 2, \cdots, n)$$

where p_i and q_i denote respectively the price and the quantity of the ith of n commodities, and putting

$$(1.2) \qquad p_{ij} = \frac{\partial p_i}{\partial q_j},$$

the conditions

$$(1.3) \qquad p_{ij} = p_{ji},$$

together with the inequalities on the Jacobian determinants

$$(1.4) \quad \begin{aligned} &\frac{\partial p_i}{\partial q_i} < 0, \quad \frac{\partial(p_i, p_j)}{\partial(q_i, q_i)} > 0, \quad \frac{\partial(p_i, p_j, p_k)}{\partial(q_i, q_j, q_k)} < 0, \cdots, \text{ and} \\ &\frac{\partial q_i}{\partial p_i} < 0, \quad \frac{\partial(q_i, q_j)}{\partial(p_i, p_j)} > 0, \quad \frac{\partial(q_i, q_j, q_k)}{\partial(p_i, p_j, p_k)} < 0, \cdots, \end{aligned}$$

which generalize the condition that a demand curve shall decline, have been deduced[1] on the assumption that the buyers are entrepreneurs,

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[1] Harold Hotelling, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions," *Journal of Political Economy*, XL (1932), 571–616, especially p. 590. An interesting statistical study involving a test of the conditions (1.3) and the equivalent conditions

$$\frac{\partial q_j}{\partial p_i} = \frac{\partial q_i}{\partial p_j}$$

is contained in a paper by Henry Schultz: "Interrelations of Demand," *Journal of Political Economy*, Vol. XLI (August, 1933). Among the four commodities, barley, corn, hay, and oats, six equations of each of these types are tested. In most cases the differences between the left and right members as calculated statistically appear to be well within the discrepancies to be expected on the basis of the standard errors, though one of the six differences (hay-oats) is too great to be ascribed to chance alone. Schultz' results as a whole seem to confirm the

or others adjusting their purchases for maximum profit, in such a way that their purchases could be increased without effective limit if they should lead to sufficiently increased profits. Similar conditions were deduced for supply functions. In case a buyer's budget is limited, it was pointed out that his demand functions will not, in general, satisfy the conditions (1.3), but will satisfy

$$(1.5) \qquad p_i(p_{jk} - p_{kj}) + p_j(p_{ki} - p_{ik}) + p_k(p_{ji} - p_{ji}) = 0.$$

We now inquire what inequalities are to replace (1.4) in case of a limited budget. The results obtained are applicable not only to the purchases of individuals, where we are dealing with commodities for actual consumption, and forming a substantial part or all of their expenditures, but likewise to departments of corporations or governments, which often have their budgets fixed in advance, with little opportunity for variation on account of price changes within a budget period.

Results similar to those of the present paper may be obtained for supply functions of a certain character. Thus, a subsistence farmer, who must get a certain cash return to meet taxes and necessary money expenditures, might be treated as having a negative budget, which must be filled by some selection of cash crops which the farmer will seek to produce with a minimum of disutility. Conditions on such supply functions may be derived from the results that we shall obtain by changing the sign of m, the amount of the budget. Negative quantities may be regarded as sales, positive quantities as purchases. The prices, however, will never be negative.

II

As a preliminary, we shall retrace the former argument, with more detailed attention to the process of passing from the demand functions of individuals to the total demand functions, which can be examined statistically. We suppose there are N competing buyers of n commodities, and use, in this section only, a system of subscripts in which Greek letters denote individual buyers, while Latin letters correspond to the commodities. Suppose that the αth buyer, by purchasing quantities

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applicability to these commodities of the integrability conditions. It must be remembered that even though a single discrepancy may be judged significant when it exceeds double its standard error, still, among six, it is quite probable that one will fall beyond this limit. The standard errors of the differences tested are somewhat difficult to determine, since the partial derivatives are taken as regression coefficients found from separate least-square solutions, between which the residuals, and therefore the solutions, are correlated. No exact theory exists for dealing with this situation with full statistical efficiency.

$q_{1\alpha}, q_{2\alpha}, \cdots, q_{n\alpha}$ of the several commodities can produce and sell goods that will bring him a gross money income u_α. This gross income we write as a function of the $q_{i\alpha}$:

$$u_\alpha = u_\alpha(q_{1\alpha}, \cdots, q_{n\alpha}).$$

The net income of this buyer is then

$$(2.1) \qquad \pi_\alpha = u_\alpha - p_1 q_{1\alpha} - p_2 q_{2\alpha} - \cdots - p_n q_{n\alpha}.$$

This is to be made a maximum. The buyer is assumed to regard the n prices as fixed by the market, independent of his purchases. Hence,

$$(2.2) \qquad \frac{\partial \pi_\alpha}{\partial q_{i\alpha}} = \frac{\partial u_\alpha}{\partial q_{i\alpha}} - p_i = 0.$$

Defining $f_{i\alpha}$ as

$$(2.3) \qquad f_{i\alpha} = f_{i\alpha}(q_{1\alpha}, q_{2\alpha}, \cdots, q_{n\alpha}) = \frac{\partial u_\alpha}{\partial q_{i\alpha}},$$

we thus have

$$(2.4) \qquad p_i = f_{i\alpha}(q_{1\alpha}, \cdots, q_{n\alpha}) \qquad (i = 1, 2, \cdots, n)$$

as the demand functions for this buyer. Evidently, on account of (2.3), they satisfy the integrability conditions

$$(2.5) \qquad \frac{\partial f_{i\alpha}}{\partial q_{j\alpha}} = \frac{\partial f_{j\alpha}}{\partial q_{i\alpha}},$$

which may be written more briefly in the form

$$(2.6) \qquad \frac{\partial p_i}{\partial q_{j\alpha}} = \frac{\partial p_j}{\partial q_{i\alpha}}.$$

Also, the n equations (2.4) may be solved to give the quantities that this buyer will purchase at given prices, in the form

$$(2.7) \qquad q_{i\alpha} = F_{i\alpha}(p_1, p_2, \cdots, p_n).$$

These last functions satisfy integrability conditions similar to (2.6). Indeed, differentiating p_i with respect to p_k,

$$(2.8) \qquad \sum_j \frac{\partial p_i}{\partial q_{j\alpha}} \frac{\partial q_{j\alpha}}{\partial p_k} = \delta_{ik},$$

where $\delta_{ik} = 0$ if $i \neq k$, but equals unity if $i = k$. Holding k fixed but letting i vary from 1 to n, we have in (2.8) n equations in the n unknowns

$$\frac{\partial q_{1\alpha}}{\partial p_k}, \frac{\partial q_{2\alpha}}{\partial p_k}, \cdots, \frac{\partial q_{n\alpha}}{\partial p_k}.$$

Putting H_α for the determinant in which the element in the ith row and jth column is $\partial p_i / \partial q_{j\alpha}$, we have as the solution,

$$(2.9) \qquad \frac{\partial q_{j\alpha}}{\partial p_k} = \frac{\text{cofactor of } \dfrac{\partial p_k}{\partial q_{j\alpha}} \text{ in } H_\alpha}{H_\alpha},$$

a result familiar in implicit function theory. But from (2.6) it follows that H_α is a symmetrical determinant, whence, from (2.9),

$$(2.10) \qquad \frac{\partial q_{j\alpha}}{\partial p_k} = \frac{\partial q_{k\alpha}}{\partial p_j}.$$

Now letting

$$(2.11) \qquad q_i = F_i(p_1, \cdots, p_n) \qquad (i = 1, 2, \cdots, n)$$

be the total demand functions for the whole aggregate of buyers, we have

$$(2.12) \qquad q_i = \sum_\alpha q_{j\alpha}.$$

Hence, summing (2.10) with respect to α,

$$(2.13) \qquad \frac{\partial q_j}{\partial p_k} = \frac{\partial q_k}{\partial p_j}.$$

Furthermore, if we solve (2.11) for the p_i, thus writing the total demand functions in the form (1.1), these functions must satisfy

$$(2.14) \qquad \frac{\partial p_i}{\partial q_j} = \frac{\partial p_j}{\partial q_i},$$

in accordance with the same argument used in deducing (2.10) from (2.6). These are the conditions (1.3).

When the foregoing first-order conditions are satisfied, the buyer's profit π_α will be a maximum if the second derivatives of π_α, which are the same as those of u_α, are the coefficients of a negative definite quadratic form. With the notation used in (2.6), this means that

$$(2.15) \qquad \sum_i \sum_j \frac{\partial p_i}{\partial q_{j\alpha}} x_i x_j$$

must be negative definite. Since the conjugate of a negative definite form is negative definite, we have from (2.9) that

$$(2.16) \qquad \sum \sum \frac{\partial q_{j\alpha}}{\partial p_i} x_i x_j$$

is likewise negative for all values of $x_1, x_2, \cdots, x_n$ which do not all vanish. Upon summing (2.16) with respect to α and using (2.12), we find that

$$(2.17) \qquad \sum \sum \frac{\partial q_j}{\partial p_i} x_i x_j$$

is likewise negative definite. This conclusion is equivalent to the conditions on the total demand functions (2.11),

$$(2.18) \qquad \frac{\partial q_i}{\partial p_i} < 0, \quad \frac{\partial(q_i, q_j)}{\partial(p_i, p_j)} > 0, \quad \frac{\partial(q_i, q_j, q_k)}{\partial(p_i, p_j, p_k)} < 0, \cdots.$$

From the negative definite character of (2.17) follows that of its conjugate, from which follow the conditions (1.4).

For supply functions a similar argument leads to positive definite forms, with results which differ from the foregoing formulae only in that all the inequality signs are $>$.

III

Turning now to the case of a buyer with a limited budget m, which will be assumed positive, we refer to the concept of a utility function ϕ which this person desires to make a maximum. This function ϕ of the quantities purchased is of the nature of a rank index, distinguishing one "indifference surface" from another, and need not be regarded as uniquely defined. Indeed, any other function ψ of the quantities, provided only that ψ is an increasing function of ϕ with a sufficient number of continuous derivatives, will be equally satisfactory. From this it will follow that our results must have an invariantive character, in that they may be expressed in terms either of ϕ or of ψ, and each of these statements must be a logical consequence of the other.

We shall hereafter omit the additional subscript α used in the last section to distinguish among individuals, and denote the quantities a particular person buys merely by $q_1, \cdots, q_n$.

Subject to the condition

$$(3.1) \qquad \sum_{i=1}^{n} p_i q_i = m,$$

the function

$$(3.2) \qquad \phi = \phi(q_1, q_2, \cdots, q_n)$$

is to be a maximum. The first-order conditions for this are

$$(3.3) \qquad \phi_i = \lambda p_i, \qquad (i = 1, 2, \cdots, n)$$

where $\phi_i = \partial \phi / \partial q_i$, and the Lagrange multiplier λ is the "marginal utility of money," and is essentially positive, since both p_i and ϕ_i are positive, the latter because ϕ is assumed to be an increasing function of the quantities.

The demand functions are found by eliminating λ between (3.1) and (3.3), in the form

$$(3.4) \qquad p_i = \frac{m\phi_i}{\sum_j \phi_j q_j}.$$

Denoting differentiation with respect to q_j by an added subscript j, we have from (3.3).

$$(3.5) \qquad \phi_{ij} = \lambda p_{ij} + \lambda_j p_i.$$

Since $\phi_{ij} = \phi_{ji}$, the condition (1.5) may be obtained from (3.5) by interchanging i and j and subtracting, multiplying by p_k, permuting i, j, and k cyclically, and adding the three equations thus obtained.

The second-order condition[2] for a maximum of ϕ subject to the constraint (3.1) is that each root of the equation in σ,

$$(3.6) \qquad \begin{vmatrix} \phi_{11} - \sigma & \phi_{12} & \cdots & \phi_{1n} & p_1 \\ \phi_{21} & \phi_{22} - \sigma & \cdots & \phi_{2n} & p_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \phi_{n1} & \phi_{n2} & \cdots \phi_{nn} - \sigma & p_n \\ p_1 & p_2 & \cdots & p_n & 0 \end{vmatrix} = 0,$$

shall be negative. The roots are known all to be real, on account of the symmetry of the determinant and the reality of the ϕ_{ij} and p_i.

If a function of n variables possesses a maximum for a certain set of values of these variables, whether or not there are subsidiary conditions, and if some of these variables are held fixed at their maximizing values while the others vary, the function must under this new type of variation still be a maximum for the same values as before. It follows that the second-order conditions applicable to the whole set of varia-

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[2] Harris Hancock: *Theory of Maxima and Minima* (1917), pp. 115–116.

bles are likewise applicable to every subset of them, so long as any degree of freedom remains. In particular, rows other than the last, and the like-ordered columns, may be deleted from (3.6), and the resulting equations must have all their roots negative. At least three rows must, however, remain in the reduced determinant in order that it may actually involve σ and thus express a condition.

In (3.6) substitute the values (3.5) of the second derivatives. Then from the jth column subtract the product of λ_j by the last column. Divide each column but the last by λ, and multiply the last row by λ. Upon putting

$$\rho = \sigma/\lambda,$$

so that ρ has the same sign as λ, this gives the result that all the roots of

$$(3.7) \qquad \begin{vmatrix} p_{11} - \rho & p_{12} & \cdots & p_{1n} & p_1 \\ p_{21} & p_{22} - \rho & \cdots & p_{2n} & p_2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ p_{n1} & p_{n2} & \cdots & p_{nn} - \rho & p_n \\ p_1 & p_2 & \cdots & p_n & 0 \end{vmatrix} = 0$$

must be negative. In like manner, all roots of all symmetrical determinantal equations formed from (3.7) by deleting rows and columns other than the last must be negative. Thus we have obtained the conditions upon the first derivatives of the demand functions (3.4) which correspond to those of the last section. These conditions may however be put in a simpler form.

Let

$$(3.8) \qquad \Delta_{12} = \begin{vmatrix} p_{11} & p_{12} & p_1 \\ p_{21} & p_{22} & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}, \qquad \Delta_{123} = \begin{vmatrix} p_{11} & p_{12} & p_{13} & p_1 \\ p_{21} & p_{22} & p_{23} & p_2 \\ p_{31} & p_{32} & p_{33} & p_3 \\ p_1 & p_2 & p_3 & 0 \end{vmatrix},$$

and likewise for other combinations of subscripts. Upon deleting from (3.7) all rows and columns other than the first two and last, and expanding in powers of ρ, the resulting equation is

$$(p_1^2 + p_2^2)\rho + \Delta_{12} = 0.$$

Since the root is to be negative, Δ_{12} must be positive, i.e., the individual demand functions for two related commodities must satisfy the inequality:

$$(3.9) \qquad \frac{\partial p_1}{\partial q_1} p_2^2 - \left(\frac{\partial p_1}{\partial q_2} + \frac{\partial p_2}{\partial q_1}\right)p_1 p_2 + \frac{\partial p_2}{\partial q_2} p_1^2 < 0.$$

Expanding the four-rowed principal minor of (3.7) which corresponds to the commodities numbered 1, 2, and 3, we have

$$- (p_1{}^2 + p_2{}^2 + p_3{}^2)\rho^2 - (\Delta_{12} + \Delta_{23} + \Delta_{13})\rho + \Delta_{123} = 0,$$

of which both roots are to be negative. This gives $\Delta_{123} < 0$, and no further conditions, since we already know that Δ_{12}, and by the same reasoning Δ_{23} and Δ_{13}, are positive. Proceeding in this way, and noting that Descartes' rule of signs requires that for negative roots all the coefficients in an equation having real roots must be of like sign, we find that our second-order conditions are equivalent to the requirement that *each of the determinants $\Delta_{ijk} \ldots$ must have the sign of $(-1)^s$, where s is the number of commodities represented in the determinant.* Thus:

$$(3.10) \qquad\qquad (-1)^s \Delta_{i_1 i_2 \cdots i_s} > 0.$$

Any of the inequalities with which we deal may in special cases be replaced by equalities, in case a maximum occurs at a point at which the representing surface has contact of specially high order with its tangent plane. However, in empirical functions, this is infinitely improbable, and we need give little attention to the possibility.

The indifference loci are $(n-1)$-dimensional hypersurfaces in an n-dimensional space in which the quantities taken by the individual are cartesian coordinates. He has no preference for one point over another on a fixed indifference locus, but desires to move from one to another of these loci in the direction of increasing quantities. However, he is constrained to remain on the hyperplane whose equation is (3.1). The collection of goods which he can buy with his limited budget, at the prices that have been fixed beyond his control and that determine the orientation of his hyperplane, will give him greatest satisfaction if he chooses their quantities equal to the coordinates of a point of tangency of the hyperplane with an indifference locus. The coordinates of this point of tangency satisfy the equations (3.4). The inequalities on the determinants $\Delta_{ijk} \cdots$ are geometrically equivalent to the requirement that, at the point of tangency chosen, the hypersurface shall be concave toward the directions of increasing quantities. The actual quantities being supposed positive, this means that the hypersurface must be convex to the origin at any point that can actually represent the purchases of an individual who makes the best of his budget. In particular, for two commodities, we have a system of indifference curves, one of which will be tangent to the straight line determined by the prices and the amount of money to be spent. For maximum utility this indifference curve must, at the point of contact, bend away from the straight line in the direction of increasing quantities.

If indifference curves for purchases be thought of as possessing a wavy character, convex to the origin in some regions and concave in others, we are forced to the conclusion that it is only the portions convex to the origin that can be regarded as possessing any importance, since the others are essentially unobservable. They can be detected only by the discontinuities that may occur in demand with variation of price-ratios, leading to an abrupt jumping of a point of tangency across a chasm when the straight line is rotated. But, while such discontinuities may reveal the existence of chasms, they can never measure their depth. The concave portions of the indifference curves and their many-dimensional generalizations, if they exist, must forever remain in unmeasurable obscurity.

IV

The ordinary conception of a demand curve involves a *ceteris paribus* assumption that is peculiarly objectionable, in that there is not only the observed fact that other things do not remain equal during the variation of the price and quantity of a particular commodity, but that it is logically impossible that they do so. Indeed, for n related commodities (and all commodities are related to others), we may have the demand situation expressed in the form

$$(4.1) \qquad p_i = f_i(q_1, q_2, \cdots, q_n) \qquad (i = 1, 2, \cdots, n)$$

or in the alternative form, found by solving these equations,

$$(4.2) \qquad q_j = F_j(p_1, p_2, \cdots, p_n).$$

In tracing the relation between p_1 and q_1, the graph of the first equation in (4.1) when $q_2, \cdots, q_n$ are held constant will be different from and inconsistent with the equally valid "demand curve" obtained from (4.2) by holding $p_2, \cdots, p_n$ constant. There is a real difference between fixing the prices and fixing the quantities of the commodities related to the first, and these two fixations are not simultaneously possible.

The sets of expressions (4.1) and (4.2) may be regarded as alternative standard forms of the demand functions for n commodities related to each other but not to others. Besides these standard forms, however, there are many other possible ways of expressing the same demand relations. Indeed, any set of n independent equations among the p's and q's would suffice to define these relations. Certain of these alternative forms of demand functions will now be used to express the inequalities of the last section in another way which lends itself to the combination, in some circumstances, of these inequalities upon

individual demand functions into inequalities upon the total demand functions.

Instead of solving the whole set of equations (4.1) for the q's, let us select only s of them, denoting their indices by $i_1, i_2, \cdots, i_s$, and solve these for $q_{i_1}, q_{i_2}, \cdots, q_{i_s}$. Let the remaining $n-s$ indices be denoted by $i_{s+1}, \cdots, i_n$. The solution may be written:

$$(4.3) \quad q_\beta = F_{\beta \cdot i_{s+1} \cdots i_n}(p_{i_1}, \cdots, p_{i_s}, q_{i_{s+1}}, \cdots, q_{i_n}) \quad (\beta = i_1, \cdots, i_s).$$

We now proceed to determine the derivatives of these functions which we denote by

$$(4.4) \qquad q_{\beta\gamma \cdot i_{s+1} \cdots i_n} = \frac{\partial F_{\beta \cdot i_{s+1} \cdots i_n}}{\partial p_\gamma} \qquad (\beta, \gamma = i_1, \cdots, i_s)$$

in terms of the derivatives p_{ij} of (4.1). The derivative (4.4) differs from what we have previously called $\dfrac{\partial q_\beta}{\partial p_\gamma}$ in that the quantities instead of the prices of $n-s$ of the commodities are held constant during the differentiation.

When the expressions (4.3) are substituted for $q_{i_1}, \cdots, q_{i_s}$ in (4.1), the resulting equations are identities. Differentiating the ith of these identities with respect to p_γ, we have, since,

$$(4.5) \qquad \frac{\partial f_i}{\partial q_\beta} = p_{i\beta}, \text{ while } \frac{\partial p_i}{\partial p_\gamma} = \delta_{i\gamma},$$

where $\delta_{i\gamma}$ is unity if $i = \gamma$ but is zero if $i \neq \gamma$,

$$(4.6) \qquad \sum_{\beta = i_1}^{i_s} p_{i\beta} q_{\beta\gamma \cdot i_{s+1} \cdots i_n} = \delta_{i\gamma}.$$

Keeping γ fixed as one of the numbers $i_1, \cdots, i_s$, and letting i vary through this set of numbers, we have in (4.6) a set of s linear equations in the unknowns (4.4). Denote the determinant of the coefficients by

$$(4.7) \qquad J_{i_1 i_2 \cdots i_s} = \begin{vmatrix} p_{i_1 i_1} & p_{i_1 i_2} & \cdots & p_{i_1 i_s} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{i_s i_1} & \cdot & \cdots & p_{i_s i_s} \end{vmatrix}.$$

Then the solution of (4.6) is:

$$(4.8) \qquad q_{\beta\gamma \cdot i_{s+1} \cdots i_n} = \frac{\text{cofactor of } p_{\gamma\beta} \text{ in } J_{i_1 \cdots i_s}}{J_{i_1 \cdots i}}.$$

Now expand an s-rowed determinant of the type (3.8) with reference to its last row and last column. With the help of (4.8) and (3.10) the result may be written:

$$(4.9) \qquad (-1)^{s+1}\Delta_{i_1\ldots i_s} = J_{i_1\ldots i_s} \sum \sum p_\beta p_\gamma q_{\beta\gamma\cdot i_{s+1}\ldots i_n} < 0.$$

In this expression the sums are with respect to β and γ, which range over values $i_1, i_2, \cdots, i_s$.

If, for example, we take $s=2$, we have from (4.9)

$$-\Delta_{12} = J_{12}\left[p_1^2 q_{11\cdot 34\ldots n} + p_1 p_2 (q_{12\cdot 3\ldots n} + q_{21\cdot 3\ldots n}) + p_2^2 q_{22\cdot 3\ldots n}\right] < 0,$$

where

$$J_{12} = p_{11}p_{22} - p_{12}p_{21}.$$

V.

Now consider the additional hypothesis that, for each buyer in the market, the determinant (4.7) has the sign of $(-1)^s$. In this case (4.9) takes the form

$$(5.1) \qquad\qquad (-1)^s \sum \sum p_\beta p_\gamma q_{\beta\gamma\cdot i_{s+1}\ldots i_n} < 0,$$

a relation valid for each buyer. Upon summing for all the buyers, since the p's are the same for all, and since the derivative $q_{\beta\gamma\cdot i_{s+1}\ldots i_n}$ of the total quantity bought is the sum of such derivatives of quantities bought by individuals, we find that *the inequalities* (5.1) *hold when applied to the market as a whole*, provided the hypothesis regarding the signs of the determinants is satisfied.

In this case, if we assume also that the determinant $J_{i_1\ldots i_s}$ pertaining to the whole market also has the sign of $(-1)^s$, we can pass back from (5.1) to (4.9), and thence to (3.10). Thus, from the hypotheses mentioned, it follows that s-rowed determinants such as (3.8) will have the same sign for the total demand functions as for the individual demand functions.

That this assumption regarding the signs of the determinants (4.7) may reasonably be expected to be satisfied in an extended class of cases is indicated by the argument of II, since the conclusion (1.4) is equivalent to the assumption here made. The fact that (1.4) consists of inequalities rather than equations is enough to show that its validity extends beyond the scope of the assumptions of II from which (1.4) was derived. Thus there is considerable reason to expect that the inequalities (3.10) and (4.9) can be applied to total demand functions. This inference is strengthened by the further observation that it is not necessary to the conclusion that *all* the buyers should have their determinants (4.7) of the signs assumed, but only a preponderance of

them in the weighted average formed by adding together the second members of (4.9).

One further argument adds weight to our conclusion that the inequalities we have derived for individual demand functions can with considerable probability be supposed to hold for total demand functions. For the case $s = n$, that is, the case involving all the commodities among which the budget is to be apportioned, we can actually prove that the determinant (4.7) has, for each individual, the sign assumed for it.

VI

Differentiating the individual demand function (3.4), which may be written

$$p_i = m \frac{\phi_i}{\sum \phi_k q_k},$$

with respect to q_j, we obtain

$$(6.1) \qquad p_{ij} = \frac{\partial p_i}{\partial q_j} = m \frac{\phi_{ij} \sum \phi_k q_k - \phi_i(\phi_j + \sum \phi_{jk} q_k)}{(\sum \phi_k q_k)^2}.$$

From (3.1) and (3.3),

$$(6.2) \qquad \sum \phi_k q_k = \lambda m.$$

Substituting this in (6.1) gives

$$(6.3) \qquad p_{ij} = \frac{\lambda m \phi_{ij} - \phi_i(\phi_j + \sum \phi_{jk} q_k)}{\lambda^2 m}.$$

Substituting this in the n-rowed determinant of the p_{ij} and multiplying each row and column by $\lambda^2 m$, we have

$$(6.4) \quad \begin{aligned} &\lambda^{2n} m^n J_{12\cdots n} \\ &= \begin{vmatrix} \lambda m \phi_{11} - \phi_1(\phi_1 + \sum \phi_{1k} q_k) & \cdots & \lambda m \phi_{1n} - \phi_1(\phi_n + \sum \phi_{nk} q_k) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \lambda m \phi_{n1} - \phi_n(\phi_1 + \sum \phi_{1k} q_k) & \cdots & \lambda m \phi_{nn} - \phi_n(\phi_n + \sum \phi_{nk} q_k) \end{vmatrix}. \end{aligned}$$

Let us border this determinant with zeros on the right, with unity in the lower right-hand corner, and with

$$\phi_j + \sum \phi_{jk} q_k$$

at the foot of the jth column ($j = 1, 2, \cdots, n$). In the bordered determinant, add to the ith row the product of ϕ_i by the last row ($i = 1, \cdots, n$). Thus we obtain from (6.4)

$$\lambda^{2n}m^n J_{1\cdots n} = \begin{vmatrix} \lambda m\phi_{11} & \cdots & \lambda m\phi_{1n} & \phi_1 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda m\phi_{n1} & \cdots & \lambda m\phi_{nn} & \phi_n \\ \phi_1 + \sum \phi_{1k}q_k & \cdots & \phi_n + \sum \phi_{nk}q_k & 1 \end{vmatrix}$$

Multiply the last column of this determinant by λm and divide each of the first n rows by the same quantity. Then subtract from the last row the product of q_k by the kth row ($k=1, \cdots, n$). The element in the lower right corner vanishes in accordance with (6.2), and we have

$$(6.5) \qquad \lambda^{n+1}m J_{1\cdots n} = \begin{vmatrix} \phi_{11} & \cdots & \phi_{1n} & \phi_1 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \cdots & \phi_{nn} & \phi_n \\ \phi_1 & \cdots & \phi_n & 0 \end{vmatrix}$$

Let us now revert to the equation (3.6), whose roots must all be negative. In accordance with Descartes' rule of signs and the reality of the roots, this requires that the coefficients of all powers of σ be of the same sign. In particular, the coefficient of the highest power of σ must have the same sign as the term independent of σ. But the former is

$$(-1)^n(p_1^2 + p_2^2 + \cdots + p_n^2),$$

while the term independent of σ, obtained by putting $\sigma = 0$ in (3.6), is simply λ^2 times the determinant in (6.5), by (3.3). This establishes the desired conclusion,

$$(-1)^n J_{12\cdots n} > 0,$$

since $m > 0$.

It can now be shown that *all* the Jacobian determinants (4.7) of order two or more, calculated from individual demand functions, and not simply those involving all the commodities in the system, have the signs assumed at the beginning of the fifth section above. This is not an independent assumption, but can be proved by the reasoning used in the last section. It is only necessary to replace m by the amount of money spent on the subgroup of commodities considered, and to change slightly the meaning of λ, so that it is a function of these rather than of all the quantities. The legitimacy of this follows from the remark in the paragraph following (3.6). This greatly strengthens the probability that the inequalities proved for individual buyers can all be applied to total demand functions, and definitely proves that (5.1) can thus be applied.

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BEMERKUNGEN ZU EINER THEORIE
DER RAUMWIRTSCHAFT

Von Erich Schneider

In meiner Arbeit "Preisbildung und Preispolitik unter Berücksichtigung der geographischen Verteilung von Produzenten und Verbrauchern"[1] habe ich untersucht, welche Modifikationen die Gesetze der Preisbildung auf einem vollkommenen Markte erfahren, wenn die Vollkommenheit des Marktes durch das Vorhandensein von räumlichen Trennungen zwischen Produzenten und Konsumenten gestört wird. Die Analyse wurde unter der Annahme durchgeführt, dass die Produktionszentren und Verbrauchszentren in einem gegebenen Wirtschaftsraum in gegebener Weise diskontinuierlich verteilt sind. Wir wollen in dieser Abhandlung die Voraussetzung der diskontinuierlichen Verteilung der Verbrauchszentren fallen lassen und die Betrachtung auf den Fall ausdehnen, wo ein als *kontinuierlich* betrachtetes Verbrauchsgebiet von einer Reihe von *diskontinuierlich* verteilten Produktionszentren versorgt wird.

In unserer Analyse werden die Transportkosten für die Mengeneinheit eines Gutes von den Produktionszentren zu den Verbrauchszentren eine wesentliche Rolle spielen. Diese Transportkosten lassen sich im allgemeinen nicht durch einen einfachen analytischen Ausdruck darstellen. Aus Gründen der Einfachheit können wir jedoch ohne grosse Einschränkung der Allgemeinheit der abzuleitenden Ergebnisse die Transportkosten für die Mengeneinheit eines Gutes als proportional mit der Entfernung vom Produktionszentrum annehmen. Wir betrachten also, sofern nichts anderes bemerkt wird, die Transportkosten für die Mengen- und Entfernungseinheit d.h. den Frachtsatz als konstant. Um die Rechnungen zu vereinfachen, soll ferner angenommen werden, dass der Transport von einem Produktionszentrum zu einem Verbrauchszentrum auf der Verbindungsstrecke zwischen beiden erfolgt.

Wir beginnen unsere Untersuchung mit dem einfachen Falle, dass ein kontinuierliches Verbrauchsgebiet von nur einem Produktionszentrum versorgt wird und gehen dann zu dem wesentlich komplizierteren Falle über, dass das Verbrauchsgebiet von zwei und mehr Produktionszentren mit dem *gleichen* Gute versorgt wird.

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[1] *Schmollers Jahrbuch* Bd. 58 (1934), S. 257 ff.

§ 1. Ein kontinuierliches Verbrauchsgebiet wird von einem einzigen Produzenten versorgt.

Bezeichnen π den Preis des Gutes im Produktionszentrum, also den Preis "ab Werk" und f den Frachtsatz, so beträgt der Preis p_z des Gutes in der Entfernung z vom Produktionszentrum:

$$(1) \qquad\qquad p_z = \pi + f \cdot z.$$

Für alle Verbraucher längs der Peripherie des Kreises mit dem Radius z um das Produktionszentrum als Mittelpunkt ist also das Gut zum Preise p_z erhältlich. Diesem Preise entspricht auf der Kreisperipherie eine eindeutig bestimmte Nachfrage nach dem betreffenden Gute. Da p_z mit wachsendem z wächst, wird diese Nachfrage, sobald z einen gewissen Wert überschreitet, verschwinden. Diese Entfernung, bei der jegliche Nachfrage nach dem betreffenden Gute in dem Gebiete verschwindet, über die hinaus also bei gegebenem Preise ab Werk kein Absatz mehr möglich ist—Launhardt[2] bezeichnet diese Entfernung treffend als die *Versendungsweite* des Gutes—lässt sich einfach angeben, wenn der Höchstpreis, bei dem jegliche Nachfrage in dem Gebiete verschwindet, bekannt ist. Bezeichnen wir diesen Höchstpreis mit p_m, so ergibt sich die Versendungsweite unmittelbar aus der Gleichung:

$$\pi + f \cdot z = p_m$$

zu:

$$(2) \qquad\qquad z = \frac{p_m - \pi}{f}.$$

Die Grösse des Absatzgebietes des Produktionszentrums ist dann gleich dem Flächeninhalt des Kreises um das Produktionszentrum als Mittelpunkt mit einem Radius der gleich der Versendungsweite ist, woraus sich ergibt, dass die Grösse des Absatzgebietes unter sonst gleichen Umständen dem *Quadrate* des Frachtsatzes *umgekehrt* proportional ist.

Wir haben bisher den Preis ab Werk als gegeben angenommen und haben jetzt zu zeigen, wie dieser Preis ab Werk selbst bestimmt wird. Dieses Problem lasst sich sehr einfach lösen, wenn wir uns des Begriffes der *transformierten Nachfragefunktion* bedienen, den ich in meiner ersten Arbeit über diesen Gegenstand (l. c.) entwickelt habe. In jedem Punkte unseres kontinuierlichen Absatzgebietes existiert eine bestimmte Preisabsatzfunktion für das von dem Produktionszentrum

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[2] Wilhelm Launhardt, *Mathematische Begründung der Volkswirtschaftslehre* (Leipzig 1885), S. 149 ff.

hergestellte Gut. Diese Funktion beschreibt, wie der Absatz des betreffenden Gutes *in diesem Punkte* des Verbrauchsgebietes mit dem Preise *in diesem Punkte* variiert. In den Ueberlegungen des Produzenten spielt aber nicht diese Funktion eine Rolle, sondern diejenige, die den Absatz in dem betreffenden Punkte des Verbrauchsgebietes in seiner Abhängigkeit von dem Preise ab Werk beschreibt. Diese Funktion kann nun leicht angegeben werden. Ist $p = F(x)$ die Nachfragefunktion für das betreffende Gut in einem Punkte des Verbrauchsgebietes, dessen Entfernung von dem Produktionszentrum z beträgt, so wird offenbar die Variation des Absatzes in dem Punkte des Verbrauchsgebietes mit dem Preise π ab Werk gegeben durch:

$$(3) \qquad\qquad \pi = F(x) - t_z,$$

wo t_z die Transportkosten für die Mengeneinheit von dem Produktionszentrum nach der betreffenden Stelle des Verbrauchsgebietes bezeichnet. Diese Funktion ist, wie man leicht sieht, nichts anderes als diejenige Preisabsatzfunktion, die gelten würde, wenn die Verbraucher nicht an der betreffenden Stelle des Verbrauchsgebietes, sondern im Produktionszentrum angesiedelt wären. Diese Funktion bezeichne ich als die von der betreffenden Stelle des Verbrauchsgebietes nach dem Produktionszentrum *transformierte Preisabsatzfunktion*. Sie unterscheidet sich von der nicht transformierten Funktion einfach um eine subtraktive Konstante, nämlich um die Transportkosten der Mengeneinheit des Gutes längs der betrachteten Strecke. Geometrisch bedeutet die Transformation der Nachfragefunktion eine Parallelverschiebung der Nachfragekurve um den Betrag der Transportkosten t_z der Mengeneinheit gegen den Anfangspunkt des Koordinatensystems.

Transformieren wir auf diese Weise die Nachfragefunktionen sämtlicher Stellen des Verbrauchsgebietes nach dem Produktionszentrum, so können wir die im gesamten Verbrauchsgebiet herrschende Nachfrage als Funktion des Preises ab Werk beschreiben und das Problem so behandeln, *als ob* sämtliche Verbraucher im Produktionszentrum angesiedelt wären, womit der Anschluss an die übliche Preistheorie erreicht ist.

Bezeichnen wir die Nachfragefunktion an einer beliebigen Stelle des Verbrauchsgebietes mit $x = H(p)$ und nehmen wir an, dass sämtliche Verbraucher die gleiche Nachfragefunktion besitzen, so wird, wie man leicht einsieht, die Gesamtnachfrage X im Verbrauchsgebiet als Funktion des Preises π ab Werk gegeben durch:

$$(4) \qquad\qquad X = 2\pi n \int_0^{(p_m - \pi)/f} H(\pi + fz) \cdot z\,dz,$$

wo n die als konstant angenommene Anzahl der Verbraucher pro
Flächeneinheit bedeutet. Wir wollen indessen die weiteren Ueber-
legungen nicht unter dieser speziellen Annahme durchführen und
deshalb die Funktion, die die Abhängigkeit der Gesamtnachfrage im
Verbrauchsgebiet von dem Preise ab Werk beschreibt, in der allge-
meinen Form

(5) $$X = X(\pi)$$

zugrundelegen.

Steht die Produktion unter der Herrschaft des erwerbswirtschaft-
lichen Prinzips, so wird, wie bekannt, der Produzent den Preis ab Werk
so stellen bzw. diejenige Produktmenge auf den Markt werfen, bei der
Grenzumsatz und Grenzkosten gleich sind. Nun ist leicht einzusehen,
dass die Grenzumsatzkurve, die der Funktion (5) entspricht, unter-
halb der Grenzumsatzkurve derjenigen Gesamtnachfragefunktion
liegt, die existieren würde, wenn sämtliche Verbraucher von vornherein
in dem Produktionszentrum angesiedelt wären. Daraus folgt, dass der
Schnittpunkt der Grenzumsatzkurve von (5) mit der Grenzkosten-
kurve des Produzenten zu einer kleineren Produktmenge und einem
höheren Preise führt als der Schnittpunkt der Grenzumsatzkurve der
Gesamtnachfragekurve, die existieren würde, wenn keine räumlichen
Trennungen zwischen Produzent und Verbrauchern vorhanden wären,
mit der Grenzkostenkurve. Eine räumliche Trennung zwischen Pro-
duzent und Verbraucher führt also stets zu einem höheren Preise im
Verbrauchszentrum und einem entsprechend geringerem Absatz, als
wenn diese Trennung nicht besteht. Da ferner, wie unmittelbar klar
ist, die räumliche Trennung zu einer Gewinnminderung des Produzen-
ten führt, ergibt sich, dass beide, Produzent und Verbraucher, durch
die räumliche Trennung einen Verlust erleiden.

§2. Ein kontinuierliches Verbrauchsgebiet wird von zwei oder mehr Produzenten mit dem gleichen Gute versorgt.

Wir dehnen jetzt unsere Untersuchung auf den Fall aus, dass ein
kontinuierliches Verbrauchsgebiet von zwei oder mehr Produzenten
versorgt wird. Erscheinungen im Preisbildungsprozess, die im Bilde der
marktvollkommenen Theorien nicht zu finden sind, ergeben sich nicht,
wenn die beiden Produzenten in dem gleichen Punkte des Verbrauchs-
gebietes angesiedelt sind. Indem wir die Nachfragefunktionen in den
einzelnen Punkten des Verbrauchsgebietes nach dem Produktionszen-
trum transformieren, können wir das Problem der Preisbildung wieder
so behandeln *als ob* sämtliche Verbraucher im Produktionszentrum
angesiedelt wären. Das Problem ist dann einfach identisch mit dem der

Preisbildung auf einem vollkommenen Markte unter polypolistischen Voraussetzungen auf der Angebotsseite und unter der Voraussetzung atomistischer Struktur der Nachfragdseite. Besonderheiten ergeben sich erst, wenn zwischen den beiden Produktionszentren eine räumliche Trennung besteht und die Entfernung der beiden Produktionszentren kleiner ist als die Summe derjenigen Versendungsweiten beider Produzenten, die sich ergeben, wenn die beiden Preise ab Werk jeweils gleich den Stückkostenminima beider Produzenten sind. Ist nämlich die Entfernung beider Produktionszentren *grösser* als die Summe dieser beiden äussersten Versendungsweiten, so sind die Absatzgebiete beider Produktionszentren, welche Höhe auch die Preise ab Werk haben mögen, stets zwei exzentrische Kreisflächen. Die Grösse des Absatzgebietes und damit die Grösse des Gewinnes eines jeden Produzenten ist dann allein eine Funktion seines eigenen Preisse ab Werk, und das Problem des Preisbildungsprozesses löst sich einfach in zwei singuläre Monopolprobleme auf. Ist dagegen die Entfernung beider Produktionszentren *kleiner* als die Summe der beiden äussersten Versendungsweiten, so werden sich die Absatzgebiete beider Produzenten im allgemeinen überschneiden. Die Grösse des Absatzgebietes eines Produzenten ist jetzt nicht nur eine Funktion seines *eigenen* Preises ab Werk, sondern gleichzeitig von der Grösse des Preises ab Werk des Konkurrenten abhängig. Jeder Produzent wird durch Variation seines Preises ab Werk nicht nur sein eigenes Absatzgebiet, sondern auch das des Konkurrenten beeinflussen, und zwar wird unter sonst gleichen Umständen eine Erhöhung (Senkung) des Preises ab Werk von seiten eines Produzenten das Absatzgebiet des Konkurrenten ausdehnen (verkleinern). Der Konkurrent wird natürlich in irgend einer Weise auf Preisvariationen des ersten Produzenten reagieren. Es entsteht ein Kampf um die Grösse der Absatzgebiete, und der Preisbildungsablauf sowohl wie die endgültige Aufteilung des Verbrauchsgebietes unter die beiden Produzenten wird wesentlich von der Art der *Strategie* abhängen, mit der dieser Kampf geführt wird. Diesen Zusammenhang zwischen der von den beiden Produzenten verfolgten Marktstrategie, dem Preisbildungsablauf und der endgültigen Aufteilung des Verbrauchsgebietes unter die beiden Produzenten haben wir jetzt aufzudecken.

I

Die erste Frage, die wir in dieser Problemkette zu beantworten haben, ist die folgende: Wie teilt sich ein als kontinuierlich besetzt gedachtes Verbrauchsgebiet unter zwei Produzenten bei *gegebenen* Preisen ab Werk auf? Dieses Problem ist zuerst von *W.* Launhardt[2] gestellt und völlig abschliessend gelöst worden. Später ist dann die

Launhardt'sche Lösung von einer Reihe von Verfassern, denen an-
scheinend die für die Theorie der Raumwirtschaft grundlegenden Ar-
beiten Launhardts unbekannt waren, wieder entdeckt worden. Wir wol-
len zunächst die ebenso einfache wie geniale geometrische Lösung des
Problems, wie Launhardt sie gegeben hat, kurz besprechen. In einem
Produktionszentren denken wir uns senkrecht zur Erdoberfläche, die wir
als eine Ebene auffassen, eine Strecke errichtet, deren Länge eine Mass-
zahl besitzt, die gleich dem Preise ab Werk in dem betreffenden Pro-
duktionszentrum ist. Ferner denken wir uns in *jedem* Punkte der Erd-
oberfläche senkrecht zur Erdoberfläche die Strecke errichtet, deren
Länge eine Masszahl besitzt, die gleich dem Preise ist, der in dem be-
treffenden Punkte des Verbrauchsgebietes für das von dem Produk-
tionszentrum bezogene Gut zu zahlen ist. Wie man leicht sieht, erfüllen
dann die oberen Endpunkte sämtlicher Strecken eine Fläche, die die
Form eines *Trichters* hat, dessen Spitze senkrecht über dem Produk-
tionszentrum liegt. Diese Trichterkonstruktion ist für den in dieser
Arbeit zu erörternden Problemkreis wie für die Behandlung aller
Fragen, die sich aus der Tatsache der räumlichen Trennung zwischen
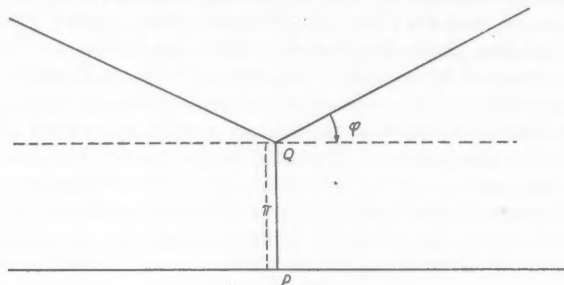
FIGUR 1
$(tg\varphi = f)$

Produktionszentren und Verbrauchszentren ergeben, von ausseror-
dentlicher Fruchtbarkeit und Tragweite. Mit ihrer Hilfe gelingt es,
Zusammenhänge zu überschauen und anschaulich zu machen, die
analytisch nur schwer oder gar nicht mehr fassbar sind. Besonders
einfach ist die Gestalt des Trichters, wenn wir die Annahme konstanten
Frachtsatzes machen. Dann ist nämlich der Trichter einfach ein
gerader Kreiskegel, dessen Spitze in einem Abstande senkrecht über
dem Produktionszentrum liegt, der gleich dem Preise ab Werk ist.
Fig. 1 zeigt einen durch das Produktionszentrum gehenden, auf der
Erdoberfläche senkrecht stehenden ebenen Schnitt durch den Trichter
für den Fall konstanten Frachtsatzes. Schneiden wir die Trichterfläche

durch eine Ebene parallel zur Erdoberfläche, so schneidet diese aus der Trichterfläche eine geschlossene Kurve aus, deren Projektion auf die Erdoberfläche die geometrische Ortslinie für alle diejenigen Punkte auf der Erdoberfläche gibt, die das Gut zum gleichen Preise von dem Produktionszentrum beziehen, zu dem Preise nämlich, der gleich der Masszahl des Abstandes der Schnittebene von der Erdoberfläche ist.

Bei gegebenem Preise ab Werk existiert für jedes Produktionszentrum ein eindeutig bestimmter Trichter. Betrachten wir jetzt zwei Produktionszentren, deren Entfernung nicht so gross ist, dass die Absatzgebiete beider Zentren für jeden beliebigen Preis ab Werk exzentrische Kreisflächen sind. Die Trichter dieser beiden Zentren werden
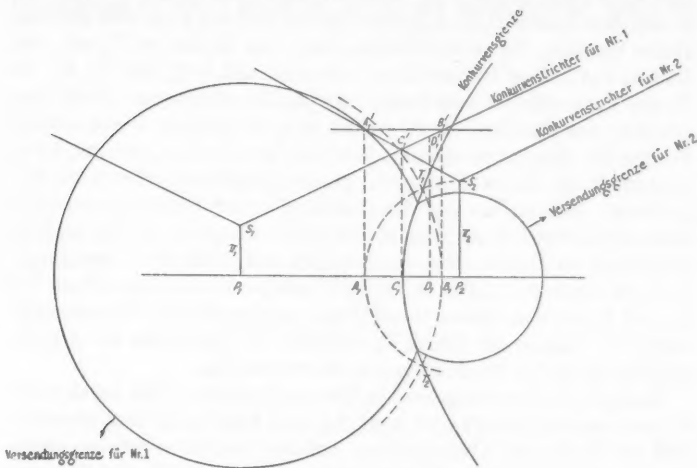
FIGUR 2

sich im allgemeinen bei gegebenen Preisen ab Werk gegenseitig durchdringen. Projizieren wir diese Durchdringungspunkte auf die Erdoberfläche, so stellen diese Projektionen alle diejenigen Punkte auf der Erdoberfläche dar, für die es gleichgültig ist, von welchem Produktionszentrum sie das betreffende Gut beziehen. Die Kurve, auf der die Projektionen der Durchdringungspunkte liegen, ist also die *Grenze der Absatzgebiete* der beiden Produktionszentren. Diese Konkurrenzgrenze, die A. Schilling[3] als *Isostante* bezeichnet hat, lässt sich auf einfache Weise konstruieren,[4] wenn die Preise ab Werk und die Abhängigkeit der Transportkosten von der Entfernung vom Produktionszentrum

[3] A. Schilling, *Die Lehre von Wirtschaften*, Berlin 1925.
[4] Drenckhahn-Schneider, *Wirtschaft und Mathematik*, Leipzig (1931), S. 39.

bekannt sind. In Fig. 2 ist diese Konstruktion für den Fall konstanten Frachtsatzes durchgeführt. Die Figur zeigt den durch die Verbindungslinie der beiden Produktionszentren P_1 und P_2 senkrecht zur Erdoberfläche gelegten ebenen Schnitt durch die beiden zu den Produktionszentren gehörenden Trichter. Ohne weiteres ist klar, dass die Projektion C_1 des Punktes C_1' auf die Strecke P_1P_2 ein Punkt der Konkurrenzgrenze ist. Um weitere Punkte der Konkurrenzgrenze zu erhalten, zeichnen wir zu P_1P_2 eine Parallele in einem Abstande, der grösser ist als C_1C_1'. Diese Parallele schneidet die Verlängerung von S_1C_1' in B_1' und die Verlängerung von S_2C_1' in A_1'. Alle Orte, die das Gut von P_1 zum Preise B_1B_1' beziehen, liegen dann auf dem Kreise um P_1 mit dem Radius P_1B_1. Alle Orte, die das Gut von P_2 zu dem gleichen Preise beziehen, liegen entsprechend auf dem Kreise um P_2 mit dem Radius P_2A_1. Diese beiden Kreise schneiden sich in T_1 und T_2. T_1 und T_2 sind dann offenbar zwei Punkte der Konkurrenzgrenze. Durch Verschieben der Parallele $A_1'B_1'$ erhält man in gleicher Weise weitere Punkte der Konkurrenzgrenze. Wie man unmittelbar erkennt, ist in unserem Falle die die Konkurrenzgrenze darstellende Kurve ein Hyperbelast. Die Grösse der Absatzgebiete beider Produktionszentren lässt sich jetzt sofort zeichnen, sobald der Höchstpreis, bei dem jegliche Nachfrage im Absatzgebiet verschwindet und damit die Versendungsweite bekannt sind. Ist D_1D_1' dieser Höchstpreis, so haben wir nur um P_1 und P_2 mit den diesem Höchstpreise entsprechenden Versendungsweiten als Radien die Kreise zu zeichnen, um die Grösse der Absatzgebiete der beiden Produktionszentren zu erhalten.

Dass die Konkurrenzgrenze in dem vorliegenden Falle durch einen Hyperbelast dargestellt wird, lässt sich auch leicht analytisch einsehen. Soll ein Punkt des Absatzgebietes auf der Konkurrenzgrenze liegen, so muss für seine Abstände z_1 und z_2 von den beiden Produktionszentren P_1 und P_2 offenbar die Beziehung gelten:

$$(6) \qquad \pi_1 + fz_1 = \pi_2 + fz_2,$$

die sich auch in der Form schreiben lässt:

$$(7) \qquad z_1 - z_2 = \frac{\pi_2 - \pi_1}{f}.$$

Für alle auf der Konkurrenzgrenze liegenden Punkte des Absatzgebietes muss also die Differenz ihrer Entfernungen von den beiden Produktionszentren konstant sein. Die Konkurrenzgrenze muss also eine Hyperbel sein, "deren hohle Seite dem teureren Ursprungsorte zugekehrt ist."[5]

[5] W. Launhardt, *loc. cit.* S. 158. s. a. A. Marshall, *Principles* (7. Aufl. 1916), S. 442, Note 1.

Sind die Preise ab Werk in den beiden Produktionszentren gleich, so geht die Bedingung (7) über in:

$$(8) \qquad\qquad z_1 = z_2,$$

was bedeutet, dass in diesem einfachen Falle die Konkurrenzgrenze die Mittelsenkrechte der Verbindungslinie der beiden Produktionszentren ist. Fig. 3 zeigt die Aufteilung des Absatzgebietes unter drei, nicht auf einer Geraden liegende Produktionszentren unter der Annahme, dass die Preise ab Werk in sämtlichen Produktionszentren gleich sind und der Frachtsatz konstant ist.

Die analytische Bestimmung der Gleichung der Konkurrenzgrenze ist indessen in dieser einfachen Weise nur durchführbar, wenn der
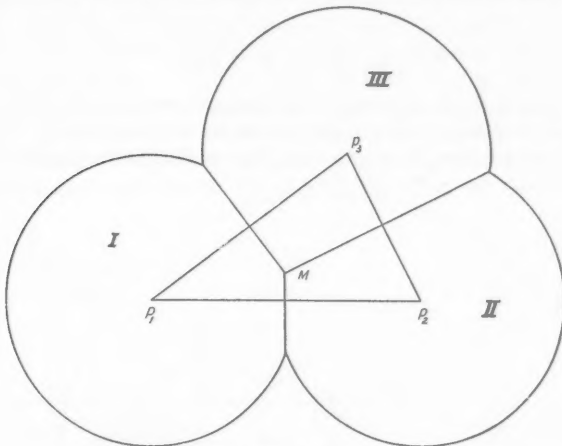
FIGUR 3

Frachtsatz konstant ist oder annähernd als konstant betrachtet werden kann. Sobald die Gütertarife so beschaffen sind, dass die Transportkosten nicht mehr der Entfernung proportional sind, was im allgemeinen der Fall ist, bereitet die rechnerische Durchführung des Problems erhebliche Schwierigkeiten, wenn sie nicht überhaupt unmöglich wird. In solchen Fällen kann allein das entwickelte geometrische Verfahren zur Bestimmung der Konkurrenzgrenze Aufschluss über die Aufteilung des Absatzgebietes unter die beiden Produzenten geben, weshalb dieses Verfahren für das gesamte Problemgebiet eine nicht unerhebliche Bedeutung besitzt.

Besonders bedeutsame und interessante Situationen ergeben sich, wenn die Annahme *gleicher* Frachsätze in den einzelnen Produktions-

zentren nicht mehr gemacht werden kann, was z.B. dann der Fall ist, wenn es sich nicht um vollkommen gleiche Güter, sondern um Ersatzgüter (etwa Steinkohle und Braunkohle) handelt oder die in den Produktionszentren hergestellten Güter zwar vollkommen gleich sind, einzelne Produktionszentren sich aber verschiedener Transportmittel (Eisenbahn, Schiff, Auto) bedienen. Um die wesentlichen Erscheinungen, die jetzt auftreten, zu studieren, betrachten wir wieder zwei Produktionszentren. Sind die voneinander verschiedenen Frachtsätze in den beiden Produktionszentren konstant, so lässt sich die Konkurrenzgrenze noch relativ einfach analytisch ermitteln. Sind f_1 und f_2 die Frachtsätze in den beiden Produktionszentren, so geht die Bedingung (6), der sämtliche Punkte der Konkurrenzgrenze genügen müssen, über in:

$$(9) \qquad \pi_1 + f_1 z_1 = \pi_2 + f_2 z_2.$$

Eine einfache Rechnung zeigt, dass die Konkurrenzgrenze, falls $\pi_1 \neq \pi_2$ ist, eine geschlossene Kurve vierten Grades ist, die dasjenige Produktionszentrum umschliesst, das den höheren Frachtsatz besitzt[6] Fig. 4
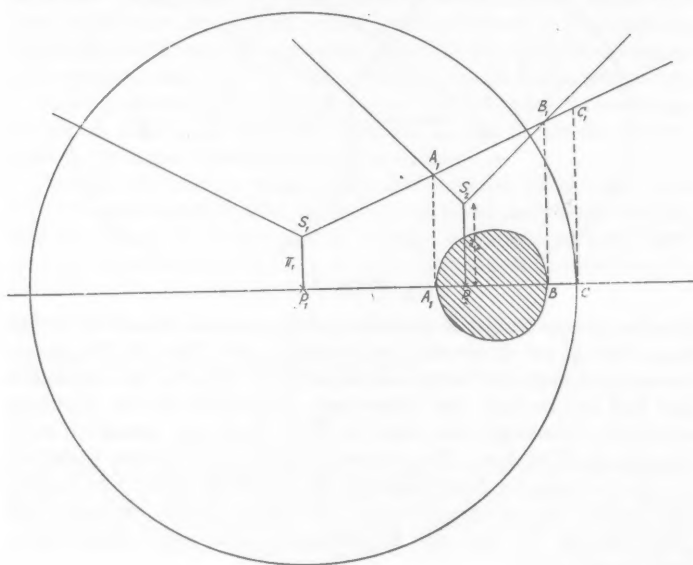
FIGUR 4

[6] W. Launhardt, *loc. cit.* S. 158.

zeigt die geometrische Konstruktion der Absatzgebiete für diesen Fall.
Ist $\pi_1 = \pi_2$, so geht die Bedingung (9) über in:

$$(10) \qquad\qquad z_1:z_2 = f_2:f_1,$$

d.h. die Entfernungen eines Punktes der Konkurrenzgrenze von den
beiden Produktionszentren müssen sich umgekehrt verhalten wie die
Frachtsätze. Die Kurve, die diese Eigenschaft besitzt, ist bekanntlich
der Apollonische Kreis, der das Zentrum mit dem höheren Frachtsatze
umschliesst.

Sind die Frachtsätze in beiden Produktionszentren verschieden, so
kann, wie bereits Launhardt erkannt hat,[7] das Produktionszentrum mit
dem höheren Preise ab Werk nie vollständig von dem Produktionszen-
trum mit dem niedrigeren Preise ab Werk aus dem Felde geschlagen
werden, wenn der Frachtsatz für das Produktionszentrum mit dem
höheren Preise ab Werk kleiner ist als der Frachtsätz für das Produk-
tionszentrum mit dem kleineren Preise ab Werk. Das gilt auch dann,
wenn die Konstellation zwischen den Preisen ab Werk so beschaffen
ist, dass das Produktionszentrum mit dem höheren Preise ab Werk im
eigenen Produktionszentrum und in einem gewissen Bereiche darüber
hinaus keinen Absatz findet. Fig. 5 zeigt diese Situation, wie sie sich
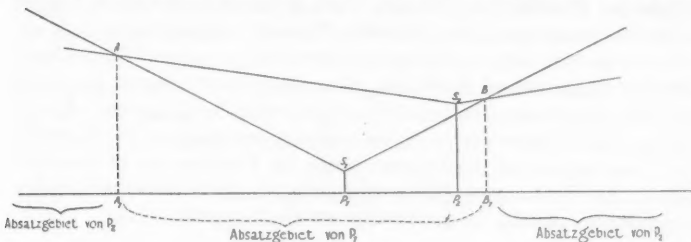
FIGUR 5

bei einem senkrecht zur Erdoberfläche durch die Verbindungslinie der
Produktionszentren gelegten ebenen Schnitt durch die Trichter dar-
stellt. Der Leser wird leicht in der Lage sein, die tatsächliche Grösse der
Absatzgebiete für die beiden Produktionszentren zu konstruieren.

Sämtliche hier angestellten Ueberlegungen lassen sich selbstver-
ständlich für mehr als zwei Produktionszentren durchführen. Das Ab-
satzgebiet eines jeden Produktionszentrums wird dann ein von Kurven-
bogen begrenztes Vieleck sein. Eine solche Konstruktion ist für
Deutschlands Steinkohlen- und Braunkohlenzentren unter Benutzung
der für 1924 geltenden Tarifsätze von Krebs[8] ausgeführt worden.

[7] W. Launhardt, *loc. cit.* S. 159–160

II

Wir haben bisher die Preise ab Werk in den einzelnen Produktionszentren als *gegeben* angenommen und unter dieser Voraussetzung die Aufteilung des Absatzgebietes unter die Produktionszentren analysiert. Wir haben uns jetzt mit der zweiten zentralen Frage in dieser Problemkette zu beschäftigen, *wie sich die Preise ab Werk in den einzelnen Produktionszentren selbst bestimmen.* Aus den bisherigen Betrachtungen hat sich ergeben, dass die Grösse des Absatzgebietes eines jeden Produzenten nicht nur von *seinem* Preise ab Werk, sondern auch von den Preisen ab Werk *sämtlicher* übrigen Produzenten abhängt. Ist aber die Grösse des Absatzgebietes eines Produzenten eine Funktion der Preise ab Werk *sämtlicher* Produzenten, so gilt das gleiche natürlich für die sich *ihm* zuwendende *Gesamtnachfrage* und damit auch für seinen *Gewinn.* Die Preise ab Werk sämtlicher Produzenten sind also zunächst als voneinander unabhängige simultane Variable bei der Bestimmung der Preise ab Werk in den einzelnen Produktionszentren zu betrachten. Um einen Einblick in den Mechanismus des Preisbildungsablaufes zu gewinnen, brauchen wir indessen die folgende Analyse nicht allgemein für n Produktionszentren durchzuführen. Alle wesentlichen Erscheinungen des Preisbildungsablaufes, die sich bei einer Situation wie der vorliegenden zeigen, treten in voller Reinheit auf und lassen sich am einfachsten überschauen, wenn wir unsere Analyse auf *zwei* Produktionszentren beschränken. (Duopol). *Aktionsparameter* unseres Problems sind dann die Preise π_1 und π_2 ab Werk der beiden Produzenten in P_1 und P_2,[9] und zwar haben wir π_1 als den Aktionsparameter des Produzenten in P_1 und π_2 als den Aktionsparameter des Produzenten in P_2 anzusehen. Die Funktionen, die die Abhängigkeit der *Gewinne* der beiden Produzenten von den beiden Preisen ab Werk angeben, seien:

$$(11) \qquad \begin{aligned} G_1 &= G_1(\pi_1,\ \pi_2) \\ G_2 &= G_2(\pi_1,\ \pi_2), \end{aligned}$$

Unter der Annahme, dass die Produktion von dem erwerbswirtschaftlichen Prinzip beherrscht wird, wird jeder der beiden Produzenten danach streben seinen Aktionsparameter so zu setzen, dass er den unter den gegebenen Verhältnissen maximalen Gewinn erzielt. Da aber sein Gewinn von den Preisen ab Werk beider Produzenten abhängt, wird jede noch so kleine Variation, die ein Produzent in *seinem* Aktions-

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[8] Krebs, "Die Frachtgrenzen der deutschen Braunkohle"; *Technik und Wirtschaft*, 1924. S. 213–218 (Berlin).

[9] Ueber den Begriff des Aktionsparameters siehe R. Frisch, "Monopole-Polypole—La notion de force dans l'Économie"; Westergaard-Festschrift; Bd. 71 (1933) der *Nationalökonomisk Tidsskrift* Kopenhagen, S. 248 ff.

parameter vornimmt, im allgemeinen eine Variation des Aktions-
parameters des anderen Produzenten als Reaktion auf die Variation
des ersten auslösen. Infolgedessen wird *jeder* Produzent, bevor er eine
kleine Variation seines Preises ab Werk vornimmt, *überlegen*, wie sein
Konkurrent auf seine geplante Variation, falls er sie realisieren würde,
reagieren wird. Das Verdienst, die fundamentale Bedeutung dieser *Vor-
stellung*, die sich ein Produzent von den *Wirkungen* einer kleinen Varia-
tion seines Aktionsparameters auf den oder die Aktionsparameter
seiner Konkurrenten macht, für den Preisbildungsablauf erkannt und
in die Theorie eingebaut zu haben, gebührt Ragnar Frisch.[10] Unsere
Analyse des Preisbildungsablaufes in dem vorliegenden speziellen Falle
ist also im wesentlichen identisch mit einer Untersuchung der Wir-
kungen jener eben charakterisierten Vorstellungen oder, in *Frisch's*
Terminologie, der *konjekturalen* Elemente des Problems.

Betrachten wir zunächst die Gewinnfunktion des ersten Produzen-
ten:

$$(12) \qquad G_1 = G_1(\pi_1, \pi_2)$$

und nehmen wir bestimmte Werte von π_1 und π_2 als Ausgangssituation
an. Glaubt nun der erste Produzent, dass eine kleine Variation seines
Preises ab Werk um den Betrag $\delta\pi_1$—wir wollen mit Frisch *vorgestellte*
(konjekturale) Variationen durch das Symbol δ andeuten, um sie von *fak-
tischen* Variationnen, die durch das Symbol ∂ bezeichnet werden, zu un-
terscheiden—eine Variation des Preises ab Werk des zweiten Produzen-
ten um den Betrag $\delta\pi_2$ zur Folge haben wird, so beträgt offenbar die
totale konjekturale Variation des Gewinnes des ersten Produzenten:

$$(13) \qquad \delta G_1 = \frac{\partial G_1}{\partial \pi_1} \delta\pi_1 + \frac{\partial G_1}{\partial \pi_2} \delta\pi_2.$$

Dividieren wir beide Seiten dieser Gleichung durch $\delta\pi_1$, so erhalten wir:

$$(14) \qquad \frac{\delta G_1}{\delta \pi_1} = \frac{\partial G_1}{\partial \pi_1} + \frac{\partial G_1}{\partial \pi_2} \cdot \frac{\delta \pi_2}{\delta \pi_1}.$$

In dieser Relation ist der Quotient $\dfrac{\delta\pi_2}{\delta\pi_1}$ ein Mass für die Grösse der
Aenderung von π_2, wie sie sich der erste Produzent als Folge einer von
ihm vorgenommenen kleinen Variation von π_1 *vorstellt*. Dieser Quotient,

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[10] R. Frisch, *loc. cit.* S. 251 ff. Zwar weist schon A. L. Bowley im *Mathematical
Groundwork of Economics* anlasslich der Behändlung des Cournot'schen Mengen-
duopols (p. 38) auf die Bedentung dieser Vorstellungen für den Systemablauf
hin, ohne jedoch darauf weiter aufzubauen.

den wir als den *konjekturalen Koeffizienten*[11] in Bezug auf den ersten Produzenten bezeichnen wollen, hängt offenbar *wesentlich* von der Ausgangssituation π_1, π_2 ab. Die Art dieser Abhängigkeit muss als ein psychologisch-technisches Datum des Problems betrachtet werden. Die Relation (14) zeigt nun, wie sich die konjekturale Variation des Gewinnes des ersten Produzenten als Folge einer von ihm vorgenommenen kleinen Variation von π_1 gleichzeitig durch Kombination dieses *konjekturalen* Datums mit den *objektiven* Daten des Problems, nämlich den partiellen Ableitungen $\dfrac{\partial G_1}{\partial \pi_1}$ und $\dfrac{\partial G_1}{\partial \pi_2}$ ergibt, die beschreiben, wie der Gewinn des ersten Produzenten *faktisch* bei einer isolierten kleinen Variation von π_1 bzw. π_2 variiert.

Entsprechende Ueberlegungen gelten für den zweiten Produzenten. Glaubt er, dass eine kleine Variation seines Preises ab Werk um den Betrag $\delta\pi_2$ eine konjekturale Variation des Preises ab Werk des ersten Produzenten um den Betrag $\delta\pi_1$ zur Folge haben wird, so hat die konjekturale Variation seines Gewinnes die Grösse

$$(15) \qquad \delta G_2 = \frac{\partial G_2}{\partial \pi_1}\, \delta\pi_1 + \frac{\partial G_2}{\partial \pi_2}\, \delta\pi_2.$$

Die Division beider Seiten dieser Gleichung durch $\delta\pi_2$ ergibt:

$$(16) \qquad \frac{\delta G_2}{\delta \pi_2} = \frac{\partial G_2}{\partial \pi_2} + \frac{\partial G_2}{\partial \pi_1} \cdot \frac{\delta \pi_1}{\delta \pi_2},$$

in der der Quotient $\delta\pi_1/\delta\pi_2$ ein Mass für die Grösse der Aenderung von π_1 ist, wie sie sich der zweite Produzent als Folge einer von ihm vorgenommenen kleinen Variation von π_2 *vorstellt*. Diesen Quotienten bezeichnen wir als den *konjekturalen Koeffizienten* in Bezug auf den zweiten Produzenten. Die Art der Abhängigkeit dieses Koeffizienten von der tatsächlichen Situation ist entsprechend wie beim ersten Produzenten ein wesentliches Datum des Problems.

Aus den konjekturalen Gewinnvariationen $\delta G_1/\delta\pi_1$ und $\delta G_2/\delta\pi_2$ der beiden Produzenten ergeben sich jeweils diejenigen Variationen der Preise ab Werk, die die beiden Produzenten in einer gegebenen Situation π_1, π_2 tatsächlich vornehmen werden. Ist die konjekturale Gewinnvariation, $\delta G_1/\delta\pi_1$ des ersten Produzenten positiv, so wird er π_1 erhöhen; ist $\delta G_1/\delta\pi_1 < o$, so wird er π_1 herabsetzen; ist endlich $\delta G_1/\delta\pi_1 = o$, so wird er keine Veränderung seines Preises ab Werk vornehmen. Entsprechendes gilt für den zweiten Produzenten. Wie gross

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[11] Frisch arbeitet in seiner zitierten Arbeit an Stelle der konjekturalen Koeffizienten mit konjekturalen Elastizitäten, was natürlich auf dasselbe hinausläuft.

die *faktisch* von den beiden Produzenten in den einzelnen Situationen π_1, π_2 vorgenommenen Variationen der Preise ab Werk sind, brauchen wir hier nicht zu untersuchen. Für unsere weiteren Betrachtungen genügt die Feststellung, dass sich das System so lange ändern muss, d.h. so lange von beiden Produzenten Variationen ihrer Preise ab Werk vorgenommen werden, bis eine Situation π_1, π_2 erreicht ist, in der *gleichzeitig die konjekturalen Gewinnvariationen beider Produzenten den Wert Null besitzen*, d.h. bis:

$$(17) \qquad \frac{\delta G_1}{\delta \pi_1} = 0 \text{ und } \frac{\delta G_2}{\delta \pi_2} = 0$$

sind. Beachten wir, dass die konjekturalen Gewinnvariationen durch die Relationen (14) und (16) gegeben werden, so muss also diejenige Kombination der Preise ab Werk beider Produzenten, bei der für beide Produzenten kein Grund mehr zu einer weiteren Aenderung ihrer Preise ab Werk vorhanden ist, den beiden Bedingungen:

$$(18) \qquad \frac{\partial G_1}{\partial \pi_1} + \frac{\partial G_1}{\partial \pi_2} \cdot \frac{\delta \pi_2}{\delta \pi_1} = 0$$

und

$$(19) \qquad \frac{\partial G_2}{\partial \pi_2} + \frac{\partial G_2}{\partial \pi_1} \cdot \frac{\delta \pi_1}{\delta \pi_2} = 0$$

genügen. Ist also überhaupt eine einen solchen Ruhezustand beschreibende Situation vorhanden, so müssen die ihr entsprechenden Preise ab Werk Lösungen dieser beiden Gleichungen sein. Es ist nun ohne weiteres klar, dass die einen Gleichgewichtszustand beschreibende Kombination, falls sie überhaupt existiert, wesentlich von der *Natur* der beiden konjekturalen Daten, also der beiden konjekturalen Koeffizienten $\delta \pi_2 / \delta \pi_1$ und $\delta \pi_1 / \delta \pi_2$ abhängt, was bedeutet, dass der Ruhezustand, den das System ev. nach einer gewissen endlichen Anzahl von Schritten erreicht, ganz wesentlich von den *Vorstellungen* abhängt, die sich jeder Produzent von den Wirkungen kleiner Aenderungen seines Aktionsparameters auf den Aktionsparameter seines Konkurrenten macht. Das ist leicht einzusehen. Wir sagten oben, dass die beiden konjekturalen Koeffizienten als *gegebene* Funktionen von π_1 und π_2 betrachtet werden müssen. Ist der Verlauf dieser Funktionen bekannt, so stellen die beiden Relationen (18) und (19) zwei Kurven in der π_1, π_2—Ebene dar. Die Gestalt dieser beiden Kurven ändert sich aber natürlich mit der Gestalt der Funktionen für die konjekturalen Koeffizienten, womit sich auch die Schnittpunkte dieser beiden Kurven, die ja die Lösungen des Systems (18) und (19) repräsentieren,

verschieben. Wir werden diese Abhängigkeit des Ruhezustandes des Systems von der Natur der konjekturalen Koeffizienten weiter unten ausführlich an einem konkreten Beispiel erläutern. Vorher wollen wir jedoch eine Reihe von besonders *wichtigen Annahmen* über die Natur der konjekturalen Koeffizienten in aller Allgemeinheit diskutieren.

I. Fall: Feder der beiden Produzenten *glaubt*, dass in *jeder* Situation π_1, π_2 eine kleine Variation seines Preises ab Werk *keine* Aenderung in dem Preise ab Werk seines Konkurrenten hervorrufen wird. Jeder Produzent betrachtet also den Preis ab Werk seines Konkurrenten als eine für ihn gegebene *Konstante*, die er nicht durch Variationen seines Preises ab Werk ändern kann. Wird die Handlungsweise eines jeden Produzenten von dieser Vorstellung beherrscht, so wollen wir mit *Frisch* sagen, jeder Produzent handele *autonom*. Wie aus der Definition der autonomen Handlungsweise folgt, findet sie mathematisch ihren Ausdruck in dem Verschwinden der konjekturalen Koeffizienten für *alle* Kombinationen π_1, π_2. Handeln *beide* Produzenten autonom, so heisst das also, dass die Relationen:

$$(20) \qquad \frac{\delta \pi_2}{\delta \pi_1} = 0 \text{ und } \frac{\delta \pi_1}{\delta \pi_2} = 0$$

für alle Kombinationen π_1, π_2 gelten. Für den Fall autonomer Handlungsweise beider Produzenten geht sonit das System (18) und (19), dem ein ev. vorhandener Ruhezustand des Systems genügen muss, über in:

$$(21) \qquad \frac{\partial G_1}{\partial \pi_1} = 0 \text{ und } \frac{\partial G_2}{\partial \pi_2} = 0.$$

Der Ablauf des Preisbildungsprozesses unter der Annahme autonomer Handlungsweise beider Produzenten, die, wie man sieht, nur als ein sehr einfacher Spezialfall der allgemeinen konjekturalen Strategie aufgefasst werden muss, ist bereits im Jahre 1885 von *W. Launhardt*[12] mit bewundernswerter Klarheit analysiert worden, weshalb wir diesen Fall zu Ehren des Mannes, dem wir die Grundlagen zu einer exakten Theorie einer Raumwirtschaft verdanken, als den Launhardt'schen Fall bezeichnen. Erst im Jahre 1929 ist die Launhardt'sche Problemstellung von Hotelling,[13] dem die Launhardt'schen Arbeiten anscheinend unbekannt waren, wieder aufgegriffen worden. Ein einfache geometrische Behandlung des Problems hat Zeuthen[14]

[12] W. Launhardt, *loc. cit.*, S. 161–164.

[13] H. Hotelling, "Stability in competition," *Economic Journal*, Bd. 41, (1929), S. 41 ff.

[14] F. Zeuthen, "Theoretical remarks on price policy. Hotelling's case with va-

in zwei bedeutsamen Arbeiten gegeben, die darüber hinaus weitere wertvolle Beitrage zu der im Entstehen begriffenen Theorie der Raumwirtschaft enthalten.

2. Fall: Ein Produzent, etwa der erste, *glaubt*, dass sein Konkurrent bei kleinen Variationen seines Preises π_1 ab Werk *autonom* handeln wird. Der erste Produzent glaubt also, dass der konjekturale Koeffizient in Bezug auf den zweiten Produzenten verschwindet und sich damit die Relation (19) auf:

$$(22) \qquad \frac{\partial G_2}{\partial \pi_2} = 0.$$

reduziert. Diese Gleichung stellt eine Kurve in der π_1, π_2—Ebene dar, die beschreibt, wie der zweite Produzent in der Vorstellung des ersten Produzenten auf Variationen von π_1 reagieren wird. *Kennt* der erste Produzent die Gewinnfunktion des zweiten Produzenten, den er als autonom handelnd betrachtet, so kann er aus (22) berechnen, wie der zweite Produzent auf Variationen seines Preises π_1 reagieren wird. Der konjekturale Koeffizient in Bezug auf den ersten Produzenten ergibt sich dann aus (22) zu:

$$(23) \qquad \frac{\eth \pi_2}{\eth \pi_1} = - \frac{\dfrac{\partial^2 G_2}{\partial \pi_2 \partial \pi_1}}{\dfrac{\partial^2 G_2}{\partial \pi_2{}^2}}.$$

Handelt nun der zweite Produzent *wirklich* autonom, so ist es klar, dass der erste Produzent seinen Konkurrenten auf dessen Reaktionsfunktion (22) durch Variation seines Preises ab Werk hin und her jagen und damit seinen Konkurrenten auf denjenigen Punkt seiner Reaktionskurve treiben kann, der für ihn, den ersten Produzenten, am vorteilhaftesten ist. Der erste Produzent beherrscht also in diesem Falle das gesamte System, weshalb wir diesen Fall als den *Herrscherfall* in Bezug auf den ersten Produzenten bezeichnen wollen. Die Bedingungen (18) und (19), denen die Gleichgewichtskombination in diesem Falle genügen müss, lauten:

$$(24) \qquad \frac{\partial G_1}{\partial \pi_1} - \frac{\dfrac{\partial^2 G_2}{\partial \pi_2 \partial \pi_1}}{\dfrac{\partial^2 G_2}{\partial \pi_2{}^2}} \cdot \frac{\partial G_1}{\partial \pi_2} = 0 \text{ und } \frac{\partial G_2}{\partial \pi_2} = 0.$$

riations," *Quarterly Journal of Econ.* (1933), Bd. 47, S. 231. Afstanden mellem Bedriftcellerne og det prispolitiske Sammenspil; Westergaard Festschrift; Sonderheft der *Nationaløkonomisk Tidsskrift*, Bd. 71, Kopenhagen, 1933.

Für den Fall, dass der *zweite* Produzent das System *beherrscht*, haben diese Bedingungen, wie man leicht einsieht, die Gestalt:

$$(25) \qquad \frac{\partial G_2}{\partial \pi_2} - \frac{\dfrac{\partial^2 G_1}{\partial \pi_1 \partial \pi_2}}{\dfrac{\partial^2 G_1}{\partial \pi_1^2}} \cdot \frac{\partial G_2}{\partial \pi_1} = 0 \quad \text{und} \quad \frac{\partial G_1}{\partial \pi_1} = 0.$$

3. Fall: Der Herrscherfall war in folgender Weise charakterisiert: (1) Ein Produzent kennt die Gewinnfunktion seines Konkurrenten und glaubt, dass sein Konkurrent eine autonome Marktstrategie verfolgt. (2) Der Konkurrent handelt tatsächlich autonom. Von Interesse ist es nun, zu untersuchen, wie sich der Preisbildungsblauf vollzieht, wenn *keiner* der beiden Produzenten *tatsächlich* autonom handelt, aber *jeder glaubt*, dass sein Konkurrent *autonom* handelt. Der erste Produzent rechnet also mit einem konjekturalen Koeffizienten:

$$(26) \qquad \frac{\delta \pi_2}{\delta \pi_1} = - \frac{\dfrac{\partial^2 G_2}{\partial \pi_2 \partial \pi_1}}{\dfrac{\partial^2 G_2}{\partial \pi_2^2}}.$$

Entsprechend rechnet der zweite Produzent mit einem konjekturalen Koeffizienten:

$$(27) \qquad \frac{\delta \pi_1}{\delta \pi_2} = - \frac{\dfrac{\partial^2 G_1}{\partial \pi_1 \partial \pi_2}}{\dfrac{\partial^2 G_1}{\partial \pi_1^2}}.$$

Die Bedingungen (18) und (19), denen die ev. vorhandene Gleichgewichtskombination π_1, π_2 genügen muss, haben somit die Gestalt:

$$\frac{\partial G_1}{\partial \pi_1} - \frac{\dfrac{\partial^2 G_2}{\partial \pi_2 \partial \pi_1}}{\dfrac{\partial^2 G_2}{\partial \pi_2^2}} \cdot \frac{\partial G_1}{\partial \pi_2} = 0$$

und

$$(29) \qquad \frac{\partial G_2}{\partial \pi_2} - \frac{\dfrac{\partial^2 G_1}{\partial \pi_1 \partial \pi_2}}{\dfrac{\partial^2 G_1}{\partial \pi_1^2}} \cdot \frac{\partial G_2}{\partial \pi_1} = 0.$$

4. Fall: Jeder der beiden Produzenten glaubt, dass eine kleine Variation seines Preises ab Werk eine *gleich grosse* Variation des Preises ab Werk seines Konkurrenten in der gleichen Richtung zur Folge haben wird (Fall konjekturaler *Parallelvorstellungen*) M.a.W. jeder Produzent rechnet mit einem *konstanten* konjekturalen Koeffizienten von der Grösse 1. Es ist also:

$$(30) \qquad \frac{\delta \pi_1}{\delta \pi_2} = 1 \text{ und } \frac{\delta \pi_2}{\delta \pi_1} = 1.$$

Die Gleichgewichtskombination π_1, π_2 muss also, wenn sie überhaupt vorhanden ist, jetzt den Bedingungen

$$(31) \qquad \frac{\partial G_1}{\partial \pi_1} + \frac{\partial G_1}{\partial \pi_2} = 0 \text{ und } \frac{\partial G_2}{\partial \pi_2} + \frac{\partial G_2}{\partial \pi_1} = 0.$$

genügen.

Die Annahme konstanter konjekturaler Koeffizienten von der Grösse 1 dürfte in den meisten Fällen für den Preisbildungsprozess im *Kleinhandel* zutreffend sein, weshalb, wie uns scheint, den Gleichungen (31) für die Erforschung der Preisbildung im Kleinhandel eine nicht unerhebliche Bedeutung zukommt.

Mit diesen 4 Fällen, mit denen wir die wichtigsten Spezialfälle der allgemeinen konjekturalen Markstrategie erschöpft zu haben glauben, wollen wir die allgemeine Diskussion abschliessen. Wie die Natur der konjekturalen Koeffizienten den Preisbildungsblauf in concreto beeinflusst, lässt sich am einfachsten an einem numerischen Beispiel überschauen. Als solches wollen wir die von Launhardt betrachtete Situation[15] wählen:

"Es handle sich um die Versorgung der Linie, welche beide im Wettkampfe befindliche Erzeugungsorte verbindet, mit einem für die Längeneinheit in seiner Menge gleichbleibenden Gute, z.B. eine Strasse mit dem erforderlichen Steinschlag" Der Verbrauch pro Längeneinheit sei also vollkommen unelastisch.[16] Sind π_1 und π_2 die Preise in den an den Endpunkten der Strasse befindlichen Erzeugungsorten, ist f der Frachtsatz und bezeichnet l die Länge der Strasse, so teilt sich, wie man leicht nachweist, die Absatzstrecke bei *gegebenen* Preisen in den Erzeugungsorten wie folgt unter die beiden Produzenten auf. Das Absatzgebiet des ersten Produzenten hat die Grösse:

$$(32) \qquad z_1 = \frac{\pi_2 - \pi_1 + l \cdot f}{2f} .$$

[15] W. Launhardt, *loc. cit.* S. 161–164.

[16] Wie man sieht sind Launhardt's Annahmen die gleichen wie diejenigen Hotelling's.

Für die Grösse des Absatzgebietes des zweiten Produzenten ergibt sich:

$$(33) \qquad z_2 = \frac{\pi_1 - \pi_2 + l \cdot f}{2f} .$$

Sehen wir von der Existenz von Produktionskosten in den beiden Produktionszentren ab, so hat die Gewinnfunktion des ersten Produzenten die Gestalt:

$$(34) \qquad G_1 = \frac{\pi_1}{2f} (\pi_2 - \pi_1 + lf).$$

Entsprechend lautet die Gewinnfunktion des zweiten Produzenten:

$$(35) \qquad G_2 = \frac{\pi_2}{2f} (\pi_1 - \pi_2 + lf).$$

Setzen wir mit Launhardt $f = 1$ und $l = 18$, so gehen die Gewinnfunktionen (34) und (35) über in:

$$(36) \qquad G_1 = \frac{\pi_1}{2} (\pi_2 - \pi_1 + 18)$$

$$(37) \qquad G_2 = \frac{\pi_2}{2} (\pi_1 - \pi_2 + 18).$$

Die konjekturalen Gewinnvariationen (18) und (19) der beiden Produzenten ergeben sich dann zu:

$$(38) \qquad \frac{\delta G_1}{\delta \pi_1} = \frac{1}{2} (\pi_2 - 2\pi_1 + 18) + \frac{\pi_1}{2} \cdot \frac{\delta \pi_2}{\delta \pi_1}$$

und

$$(39) \qquad \frac{\delta G_2}{\delta \pi_2} = \frac{1}{2} (\pi_1 - 2\pi_2 + 18) + \frac{\pi_2}{2} \cdot \frac{\delta \pi_1}{\delta \pi_2}.$$

Die möglicherweise vorhandene Gleichgewichtskombination π_1, π_2 des Systems muss also, wie früher gezeigt wurde, den beiden Bedingungen

$$(40) \qquad \pi_1 \cdot \left(2 - \frac{\delta \pi_2}{\delta \pi_1}\right) - \pi_2 = 18$$

$$(41) \qquad \pi_2 \cdot \left(2 - \frac{\delta \pi_1}{\delta \pi_2}\right) - \pi_1 = 18$$

genügen. Es ist sofort ersichtlich, dass die Lösungen dieses Gleichungs-

systems *wesentlich* von der Natur der beiden konjekturalen Koeffizienten abhängen.

1. *Fall:* Beide Produzenten handeln autonom (Launhardt'scher Fall). Die Bedingungen, denen die Gleichgewichtskombination des Systems genügen muss, lauten:

$$(42) \qquad \begin{cases} 2\pi_1 - \pi_2 = 18 \\ 2\pi_2 - \pi_1 = 18. \end{cases}$$

Ihren Lösungen $\pi_1 = 18$ und $\pi_2 = 18$ stellen die Gleichgewichtskombination des Systems dar, die, wie man leicht nachweist, stabilen Charakter besitzt. Die linearen Beziehungen (42), die in diesem speziellen Falle die *Reaktionsfunktionen* der beiden Produzenten repräsentieren, sind in Fig. 6 veranschaulicht. Die Koordinaten des Schnittpunktes S der beiden Reaktionsgraden bezeichnen den Ruhezustand des Systems. "Jeder der beiden Unternehmer würde, sobald er über oder
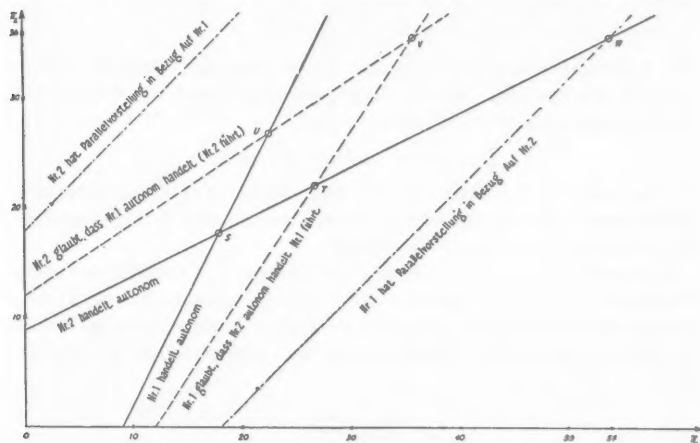
FIGUR 6

unter diesen Preis ginge, während der Gegner seinen Preis unverändert festhielte, am Gesamtgewinn verlieren."[17] Der Gewinn, den beide Produzenten in dem Ruhezustand des Systems erzielen, beträgt

$$G_1 = G_2 = 162.$$

[17] W. Launhardt, *loc. cit.* S. 162. Launhardt arbeitet in seinem Beispiel mit konstanten Stückkosten und statt der Preise ab Werk mit Gewinnzuschlägen. Da wir der Einfachheit halber von der Existenz von Kosten absehen, mussten wir den Launhardt'schen Ausdruck "Einheitsgewinn" durch "Preis" ersetzen.

2. Fall: Nehmen wir zuerst an, dass der *erste* Produzent das System in dem früher definierten Sinne beherrscht. Aus der Reaktionsfunktion des zweiten autonom handelnden Produzenten ergibt sich für seinen konjekturalen Koeffizienten der Wert:

$$(43) \qquad \frac{\delta \pi_2}{\delta \pi_1} = \frac{1}{2} .$$

In unserem Beispiel ist also die konjekturale Elastizität des Herrschers konstant. Für den zweiten Produzenten ist der konjekturale Koeffizient auf Grund der Definition der autonomen Strategie gleich Null. Die beiden Gleichungen (40) und (41) zur Bestimmung des ev. Ruhezustandes des Systems gehen somit über in:

$$(44) \qquad \begin{cases} \dfrac{3}{2}\,\pi_1 - \pi_2 = 18 \\[2mm] 2\pi_2 - \pi_1 = 18. \end{cases}$$

Die Lösungen $\pi_1 = 27$ und $\pi_2 = 22{,}5$ bestimmen den stabilen Ruhezustand des Systems. Die im Gleichgewichtszustand von den beiden Produzenten erzielten Gewinne betragen

$$G_1 = 182{,}25 \text{ und } G_2 = 253{,}125.$$

In Fig. 6 sind die den beiden Reaktionsfunktionen (44) entsprechenden Reaktionsgeraden eingezeichnet, deren Schnittpunkt T den Gleichgewichtszustand des Systems bezeichnet.

Beherrscht der *zweite* Produzent das System und handelt der erste Produzent autonom, so kehren sich die Ergebnissen um. Aus der Reaktionsfunktion des ersten, autonom handelnden Produzenten erhält man für den konjekturalen Koeffizienten des zweiten Produzenten den Wert

$$(45) \qquad \frac{\delta \pi_1}{\delta \pi_2} = \frac{1}{2} .$$

Die Gleichungen zur Bestimmung des Ruhezustandes des Systems lauten dann:

$$(46) \qquad \begin{cases} 2\,\pi_1 - \pi_2 = 18 \\[2mm] \dfrac{3}{2}\,\pi_2 - \pi_1 = 18 \end{cases}$$

deren Lösungen $\pi_1 = 22{,}5$ und $\pi_2 = 2F$ sind. Jetzt betragen die Gewinne beider Produzenten

$$G_1 = 253{,}125 \text{ und } G_2 = 182{,}25.$$

Fig. 6 zeigt die graphische Darstellung der Reaktionsfunktionen der beiden Produzenten für diesen Fall (Punkt U).

Ein Vergleich der Resultate der beiden Herrscherfälle mit dem Launhardt'schen Fall zeigt, dass jetzt beide, der Herrscher wie der autonom Handelnde einen grösseren Gewinn erzielen als im *Launhardt*'schen Fall, wo beide Produzenten autonom handeln. Dabei ist es besonders bemerkenswert, dass in den beiden Herrscherfällen autonom handelnde
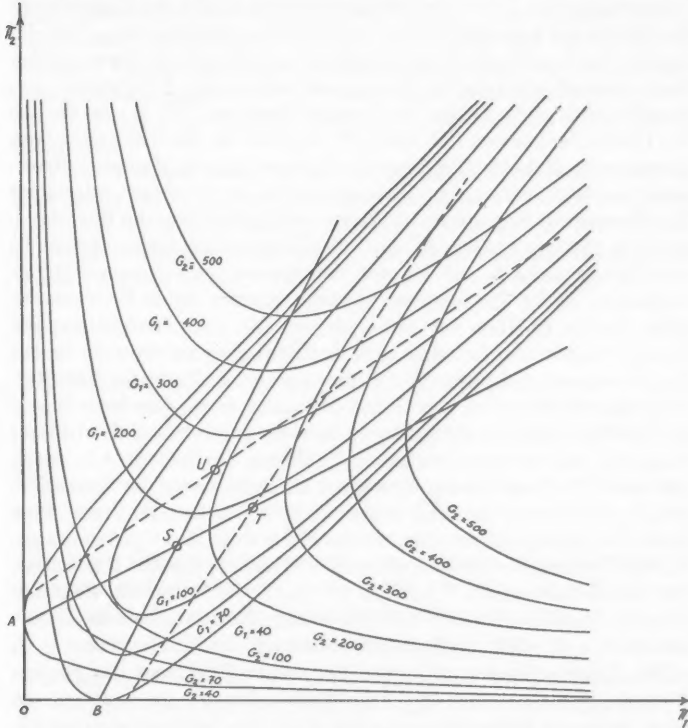
FIGUR 7

Produzent, einen *grösseren* Gewinnzuwachs gegenüber den im Launhardt'schen Falle erzielten Gewinnen erhält als der Herrscher. Diese an und für sich eigenartige Tatsache zeigt sich in noch merkwürdigerem Lichte, wenn wir sie mit den Ergebnissen vergleichen, die sich herausstellen, wenn wir das bekannte Cournot'sche Duopolproblem auf einem vollkommenen Markte sowohl unter der Annahme autonomer Stra-

tegie beider Produzenten wie auch unter der Annahme einer Herrscher-strategie von seiten eines Produzenten behandeln.[18] Hier zeigt sich nämlich, dass, sobald ein Produzent die Strategie des Herrschers verfolgt, allein der Herrscher einem höheren Gewinn erzielt (verglichen mit dem Gewinn, der sich bei beiderseitiger autonomer Handlungsweise ergibt). Der autonom handelnde Produzent erreicht einem kleineren Gewinn als den, den er bei beiderseitiger autonomer Handlungsweise erhält. Dieser eigentümliche Unterschied in den Ergebnissen, die sich bei der Lösung des Cournot'schen Duopolproblems und bei der Lösung des Launhardt-Hotelling'schen Duopolproblems herausstellen findet eine sehr einfache Erklärung, wenn wir uns der *Indifferenzkurven* für die Gewinne der beiden Produzenten bedienen. Fig. 7 zeigt die aus den Gewinnfunktionen (36) und (37) abgeleiteten Kurven konstanten Gewinnes für die beiden Produzenten. In diese Figur sind ferner die Reaktionsfunktionen (42) beider Produzenten bei gegenseitiger autonomer Handlungsweise eingezeichnet. Weiter enthält die Figur die Reaktionsfunktion für den ersten (zweiten) Produzenten im Herrscherfall. In dem Schnittpunkt S, auf den sich das System bei autonomer Handlungsweise beider Produzenten einstellt, erzielen beide Produzenten einen Gewinn in Höhe von 162 Einheiten (S). Eine Betrachtung der Figur (7) zeigt nun sofort, dass stets eine Bewegung auf einer der beiden Reaktionsgeraden AS bzw. BS im Sinne *wachsender* Preise für *beide* Produzenten vorteilhaft ist. Der Grund dafür liegt darin, dass *höher* liegenden Indifferenzkurven stets *grössere* Gewinne entsprechen als niedriger liegenden. Nun bedeutet gerade eine Führung des Systems z.B. durch den ersten Produzenten eine Bewegung des Systems auf der Reaktionsgeraden AS des zweiten Produzenten im Sinne wachsender Preise. Eine solche Bewegung steigert aber, wie die Figur zeigt, den Gewinn des geführten Produzenten stets in grösserem Masse als den des Herrschers. Der Gleichwichtspunkt T, auf den sich das System einstellt, wenn der erste Produzent führt, liegt nämlich immer noch unterhalb der Indifferenzkurve $G_1 = 200$, dagegen weit oberhalb der Indifferenzkurve G_2 = 200. Entsprechendes gilt, wenn der zweite Produzent das System führt. Dass das Resultat im Cournot'schen Duopol ein anderes, sein muss, zeigt eine Betrachtung der Fig. 8, die die Indifferenzkurven für die Gewinne der beiden Produzenten für den Fall des *Cournot'*schen Duopols enthält.[19] Im Schnittpunkt S der Reaktionsgeraden der beiden

[18] s. H. von Stackelberg, "Sulla teoria del Duopolio e del Polipolio," *Rivista Ital. di Statistica, Economia e Finanza*, Juni 1933.

[19] Die Figur gibt die Situation für den einfachen Fall einer keine Kosten verursachenden Produktion mit einer Nachfragekurve von der Form $p = 1 - x$. Wie man leicht nachrechnet, haben die Reaktionsgeraden beider Produzenten bei autonomer Handlungsweise die beiden Gleichungen

Produzenten, der bei autonomer Handlungsweise der Gleichgewichts
punkt darstellt, erzielen beide Produzenten einen Gewinn in Höhe
von 1/9 Einheiten. Beherrscht jetzt der erste Produzent das System, so
erreicht das System, wie eine einfache Rechnung zeigt, seinen Gleich-
gewichtszustand in T. Der Uebergang von S nach T führt aber, wie
die Figur zeigt, stets zu einer niedrigeren Indifferenzkurve für G_1,
dagegen zu einer höheren Indifferenzkurve für G_2. Da aber jetzt—und
das ist der wesentliche Unterschied gegenüber der Launhardt'schen
Duopolsituation—*höher* liegenden Indifferenzkurven *kleinere* Gewinne
entsprechen als niedriger liegenden, so bedeutet der Uebergang von
S nach T für den Herrscher eine Gewinnsteigerung, für den autonom
Handelnden dagegen eine Gewinnsenkung. Entsprechendes gilt bei
dem Uebergang von S nach U, wenn der zweite Produzent das System
beherrscht.

3. *Fall:* Handelt keiner der beiden Produzenten *tatsächlich* autonom,
aber *glaubt* jeder, dass sein Konkurrent autonom handelt, so ergeben
sich unter der Voraussetzung, dass jeder Produzent die Gewinnfunk-
tion seines Konkurrenten kennt, die konjekturalen Koeffizienten
beider Produzenten zu:

$$(47) \qquad \frac{\delta \pi_2}{\delta \pi_1} = \frac{\delta \pi_1}{\delta \pi_2} = \frac{1}{2}.$$

Das Gleichungssystem (40) und (41) geht dann über in:

$$(48) \qquad \begin{cases} \dfrac{3}{2}\,\pi_1 - \pi_2 = 18 \\[2mm] \dfrac{3}{2}\,\pi_2 - \pi_1 = 18. \end{cases}$$

Die den Gleichgewichtszustand des Systems charakterisierenden
Lösungen dieser beiden Gleichungen sind $\pi_1 = 36$ und $\pi_2 = 36$. Die ent-
sprechenden Gewinne der beiden Produzenten betragen:

$$G_1 = 324 \text{ und } G_2 = .324$$

Wie man sieht, ergeben sich in diesem Falle wesentlich höhere Preise
ab Werk in beiden Produktionszentren und wesentlich höhere Gewinne

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$$x_2 = \frac{1 - x_1}{2} \qquad \text{und} \qquad x_1 = \frac{1 - x_2}{2},$$

wo $x_1(x_2)$ die von dem ersten (zweiten) Produzentenn aůf den Markt geworfene
Menge bedeutet. Die Lösungen dieser beiden Gleichungen sind $x_1 = x_2 = \frac{1}{3}$. Die
diesen Mengen entsprechenden Gewinne sind $G_1 = G_2 = \frac{1}{9}$. Führt der erste Produ-
zent das System, so errechnet sich der Gleichgewichtszustand zu $x_1 = \frac{1}{2}$ und $x_2 = \frac{1}{4}$.
Der erste Produzent erzielt dann einen Gewinn $G_1 = \frac{1}{8}$, der zweite einen Gewinn
$G_2 = 1/16$. Führt der zweite Produzent das System, wird das Ergebnis umgekehrt.

Die graphische Lösung dieses Falles ist in Fig. 6 eingezeichnet (Punkt V).

4. Fall: Glaubt jeder Produzent, dass der Konkurrent auf kleine Variationen seines Preises ab Werk *parallel* reagieren wird, sind also die konjekturalen Koeffizienten für beide Produzenten gleich 1, so nimmt das Gleichungssystem (40) und (41) die Form an:

$$(49) \qquad \begin{cases} \pi_1 - \pi_2 = 18 \\ \pi_2 - \pi_1 = 18. \end{cases}$$

Diese beiden Gleichungen widersprechen sich. Das System erreicht also in diesem Falle *keinen* Gleichgewichtszustand (Fig. 6).
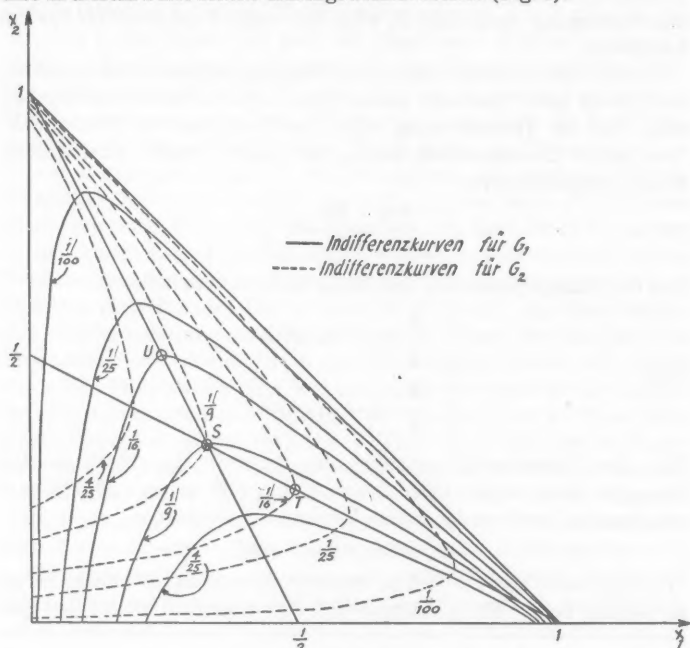
FIGUR 8

Diese Fälle, die sich noch beliebig vermehren liessen,—in Fig. 6 ist noch der Gleichgewichtspunkt für den Fall eingezeichnet, dass der erste Produzent sich in Bezug auf den zweiten Produzenten von Parallelvorstellungen leiten lässt, der zweite dagegen autonom handelt (Punkt W)—zeigen mit aller Deutlichkeit, dass die Kenntnis der Gewinnfunktionen der einzelnen Produzenten, wie sie sich objektiv aus

den Nachfragefunktionen und Kostenfunktionen ergeben, zur Analyse des Preisbildungsprozesses nicht ausreichend ist. Erst die Kombination der objektiven Gewinnfunktionen der Produzenten mit ihren konjekturalen Koeffizienten vermag uns Aufschluss über den Ablauf der Preisbildung zu geben, weshalb den konjekturalen Koeffizienten für die Untersuchung des tatsächlichen Preisbildungsblaufes zentrale Bedeutung zukommt. Welches die Natur der konjekturalen Koeffizienten in den verschiedenen konkreten Situationen ist bzw. gewesen ist, kann nur die *Erfahrung* zeigen. Es wird deshalb eine Hauptaufgabe der empirischen Forschung der Zukunft sein, uns durch eine planmässige *Beobachtung des markstrategischen Verhaltens* der Wirtschaftssubjekte in allen Zweigen der Produktion und des Handels Aufschlüsse über die tatsächliche Natur der konjekturalen Koeffizienten zu geben.

SUMMARY

The paper investigates the question of the modification of the law of price formation in a perfect market if the completeness of the market is disturbed through the existence of space separations between consumption and production. The analysis limits itself to the case where a region of consumption, considered as continuous, is provided by discontinuous distribution centers of production. The first part considers the problem of the price function for the case that a continuous region of the consumption is provided by a single production center. It is shown that this problem is solved with the help of the classical polypoly theory. If the demand curves at the different points of the market are transformed to the production center, that is, the demand is considered not as a function of the prices prevailing in the affected points of the region of consumption but of the prices in the production center. The second part considers the analysis of the complicated case where a continuous region of consumption is provided by two or more producers who are not established in the same production center. In the first section the question first treated by Launhardt is answered, how a continuous market distributes itself under two producers. In the second section the question is investigated as to how the price is determined in the separate production centers. By means of the concept of conjectural market strategy introduced by Frisch in the polypoly theory, it is shown how the pricing process depends essentially on the conjectural moment, that is, depends on the concept which the single producer forms as to the affect of his action on that of his rival.

In conclusion, the general results of this are demonstrated by the well known examples discussed by Launhardt and Hotelling.

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FAMILY BUDGETS[1]

By HANS STAEHLE

The following tables are an attempt to summarize as briefly as possible the information given in some recent family budget enquiries. The tables are far from covering all countries for which enquiries are available, nor are they complete for any single country. But it is hoped that the notes may, however, be useful as a guide to some of the more important materials. It should be emphasized that the indications given in the column on the information supplied only relate to the tables, and not to the text, given in each report. They therefore omit mentioning much valuable detail included in the more or less elaborate monographs which accompany the tables. The indications given in brackets in the first column relate to the duration of the enquiry for each family included.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
AUSTRIA 1925 (one year)	Vienna	42 workers	−2000, −2500, −3000, −3500, −4000, −4500, −5000, >5000 S. annual expenditure p. fam. −1200, −1800, >1800, S. annual income p. "man."	Expenditure (abs. and %) on main budget items according to exp. groups and for the individual families. Quantity and nutritive value of foodstuffs (actual and p. "man" p. day) for each family. Exp. on main items during each month for each family. Annual quantities of food p. "man" and p. fam. according to number of "men" in family, and acc. to annual income p. "man" (see col. 4). Elaborate study of nutritive values, and comparisons with pre-war data.	Statistische Veröffentlichungen der Wiener Kammer für Arbeiter und Angestellte: *Löhne und Lebenshaltung der Wiener Arbeiterschaft im Jahre 1925.* Wien 1928. N.B. Less detailed accounts (expenditure only) of consecutive enquiries in subsequent years, covering similar number of families, are to be found in *Wirtschafts statistisches Jahrbuch,* a yearbook published annually by the same authority.

[1] This paper represents an appendix to "Annual Survey of Statistical Information: Family Budgets," in the October, 1934, ECONOMETRICA.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
BELGIUM Apr. 1928-March 1929 (for food: 4 fortnights)	Whole country	809 workers	−200, −300, −400, >400 frs. income p. "quet" during the 4 fortnights	Very detailed expenditure (abs. and %) on, and quantities of, foodstuffs per "quet" during the 4 fortnights, acc. to income for the workers, and on the average for the employees.	XXIIᵉ Session de l'Institut International de Statistique, Londres 1934: *Résultats principaux d'une enquête sur les budgets d'ouvriers et d'employés en Belgique (1928-1929)*, by Armand Julin, La Haye 1934.
		224 salaried employees and other "small bourgeois"	none		N.B.: A bibliography is given of previous Belgian enquiries, except the one conducted by the British Board of Trade mentioned below (p. 118)
(for other exp.; one year)		116 workers	−15000, −20000, −25000, −30000, >30000 frs. annual income p. family	Expenditure p. family (abs. and %) on 12 main items of the family budget, acc. to income for the workers and on the average for the employees	
		57 salaried employees	none		
about Feb. 1932 (one month)	Brussels	19 unemployed workers	none	Expenditure p. family (abs. and %) on main budget groups and in more detail on foodstuffs (with quantities) for each family. A comparison of the individual expenditure budgets and food quantities with the 1891 and the above 1928/29 enquiries is also given.	Université Libre de Bruxelles, Institut de Sociologie Solvay: *Enquête sur les conditions de vie de chômeurs assurés: I. Le budget de 19 familles de chômeurs dans l'agglomération bruxelloise en février-mars 1932*, by G. Jacquemyns, Liège 1932.
					N.B.: Two similar enquiries have been carried out by the same author in April and May 1932 in Bruges (21 families) and Anvers (18 families).

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
BULGARIA March 1925 (one month)	Sofia	280 civil servants	−3000, −4000, −5000, −6000, −8000, >8000, Levas monthly income p. "normal family" of 4 "men"	Detailed expenditure (abs. and %) on a great number of items, and food quantities, p. family and p. "man" both according to income and for each individual family. Average prices paid are also given. A distinction is further made, in respect of expenditure, between families owning and families renting their dwelling.	Royaume de Bulgarie, Direction Générale de la Statistique: *Comptes de ménages pendant le mois de mars 1925. Livre I: ville de Sofia.* Sofia, 1927.
do.	15 provinces	{ 963 civil servants 60 artisans 82 workers	do. do. do.	Same information as above, separately for each social group (col. 3). At the end, data are given, both for expenditure and quantities of food, for each of 15 provinces, without distinction of income.	Same as above, *Livre II: Pour le Royaume,* Sofia, 1928.
June 1927–May 1928 (one year)	Whole country	173 civil servants 93 workers	−36000, −48000, −60000, −72000, −84000, −96000, >96000 Levas annual income p. "normal family" of 4 "men"	Detailed average expenditure (abs. and %) for numerous items, and quantities of foodstuffs, for each income group, p. fam. and p. "man."	Direction Générale de la Statistique, *Annuaire statistique du Royaume de Bulgarie,* XXIIIᵉ année, 1931, pp. 346–361.

CHINA

There exists a considerable number of enquiries, conducted both by individuals and scientific and other bodies. Eighty-two of these enquiries are very briefly summarised in L. K. Tao, *The Standard of Living among Chinese Workers*, China Institute of Pacific Relations, Shanghai; no date (1931?). See also Carle C. Zimmerman, "Bibliography of Studies of Family Living in Asia, Australia, New Zealand, Peru, Mexico, and The Islands of the Pacific. Preliminary Report." U. S. Department of Agriculture, Bureau of Home Economics (Stereotyped), January 1934.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
CZECHOSLOVAKIA 1928–1929 (52 weeks ending between July 1928 and June 1929)	Whole country	180 workers	−10000, −13000, −16000, −20000, −25000, −30000, −40000 Kč. annual income p. family	Expenditure p. family, p. "man," and % on main budget items, and detailed expenditure on, and quantities of, foodstuffs, p. fam. and p. "man," for the workers' families, acc. to occupations (4 groups). Same acc. to family income, separately for the workers and civil servants. Same acc. to occupational groups and within each group, acc. to family income, both for the workers and the civil servants.	*Mitteilungen des Statistischen Staatsamtes der Cechoslovakischen Republik*, Jg. XIV (1933), No. 132–137. N.B.: Similar information is available for 1927–28 in the same periodical, in Jg. XII (1931), No. 225–229; and for 1929–30 in Jg. XIV (1933), No. 138–143. These enquiries are mentioned by way of example only. Elaborations of the same data from other points of view and data reaching back as far as 1922 will be found in the same periodical. These enquiries, which cover to a large extent identical families in the consecutive periods, are being carried on every year.
		196 civil servants	−10000, −13000, −16000, −20000, −25000, −30000, −40000, −60000, >60000 Kč. annual income p. family		
DENMARK 1922 (one year)	Whole Country	80 workers 226 civil servants and 73 others	−1600, −2400, −3200, >3200 Kr. annual expenditure p. "man"	Expenditure (abs. and %) on main items, expenditure on, quantities of, foodstuffs p. family and p. "man" and territorial divisions (Copenhagen, provincial centres, rural communes). Actual expenditure on main items and quantities of foods are also given for each individual family. Average prices paid are further supplied.	Denmarks Statistik, Statistiske Meddelelser, 4. Raekke, 69. Bind, 5. Haefte; "Husholdningsregnskaber for 1922," Copenhagen 1925. N.B.: The report on a new enquiry (about 1930) is now being prepared.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
ESTONIA Nov. 1924–Oct. 1925 (for food: Oct. 1925 other items: one year)	Tallinn Narva	176 workers (food: 230 workers) 11 workers (food: 53 workers)	<6000, −8000, −10000, −12000, >12000 Mk. monthly expenditure p. family do.	Yearly expenditure p. family (abs. and %) on main items, acc. to number of persons in the family and family expenditure. Detailed exp. on, quantities and calorie value of, foodstuffs p. "man" and p. day, acc. to no. of persons in the family and family expenditure, separately for Tallinn and Narva. Housing accommodation is also studied in great detail acc. to family size and expenditure.	Bureau Central de Statistique de l'Estonie: "Budgets des familles ouvrière en 1925." Tallinn, no date.
FINLAND March 1920–Feb. 1921 (one year)	16 towns and 8 rural centres	437 workers 117 salaried employees	−5000, −7000, −9000, >9000 Mk. annual exp. p. "man" −15000, −20000, −30000, >30000 Mk. annual exp. p. fam.	Expenditure on main items p. fam. and p. "man" acc. to exp. p. family and number of "men" in the family. Detailed exp. on very numerous budget items p. fam. and p. "man" acc. to exp. p. "man" and no. of "men" in the family, and acc. to exp. p. "man" and territorial divisions. Quantities of foodstuffs p. "man" acc. to exp. p. "man" and no. of "men" in the family, and acc. to social class and territorial divisions. All tables are given separately for workers and for employees.	Finlands Officiella Statistik XXXII: Sociala Specialundersökningar V: "Levnadskostnaderna under Bokföringsperioden 1920–21." Helsinki 1925.
1928 (one year)	Helsinki	147 workers, 56 salaried employees	none	Average expenditure on main items p. fam. and p. "man," with some more details for food and a few food quantities, separately for workers and salaried employees	Social Tidskrift, published by Social ministeriet, vol. 1932, No. 12, pp. 612–624.
1928 (one year)	14 towns and 15 rural centres	581 workers, 242 salaried employees, 131 civil servants	−5000, −7500, −10000, −12500, −15000, −17500, −20000, >20000 Mk. annual exp. p. "man"	Average exp. (abs. and %) p. "normal family" of 3.4 "men" for the 3 social groups. Same acc. to exp. p. "man" for the workers and employees together. Detailed average exp. on, and quantities of, food p. "normal family" for each social group.	Social Tidskrift, as above, vol. 1933, No. 12, pp. 627–641. N.B.: A detailed report on the 1928 enquiry is to appear in the near future.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
GERMANY March 1927–Feb 1928	61 towns of all sizes	896 workers	−2500, −3000, −3600, −4300, >4300 Mk. yearly income p. fam.; −800, −1000, −1200, −1600 >1600 Mk. yearly income p. "man."	Expenditure on main items and in more detail on food (abs. and %), food quantities and their nutritive value p. fam. and p. "man," according to income p. fam. and p. "man" respectively. Expenditure is also shown according to type of dwelling (controlled or uncontrolled rents) and income; acc. to income and number of children below 15 years of age; acc. to income and the industry to which the family's head is attached. Special tables show the exp. during a month before and after a spell of unemployment or sickness for 54 families, (divided into two income groups) and the relations between housing accommodation and the number of children. Similar information is supplied separately for the employees and the civil servants. The second volume gives the detailed exp. and the food quantities for the individual families, including 37 families living in Saarbrücken.	Einzelschriften zur Statistik des Deutschen Reichs, Nr. 22: Die Lebenshaltung von 2000 Arbeiter-, Angestellten und Beamtenhaushaltungen, 2 volumes, Berlin 1932.
		498 civil servants	−3000, −3600, −4300, −5100, −6100, −7300, −10000, >10000 Mk. yearly income p. fam.; −1000, −1200, −1500, −1800, −2200, −2600, −3100, >3100 Mk. yearly income p. "man."		
		542 salaried employees	−3000, −3800, −4300, −5100, −6100, >6100 Mk. yearly income p. fam.; −1000, −1200, −1600, −1800, −2200, >2200 Mk. yearly income p. "man."		
1925 (one year)	Hamburg	67 workers and 13 salaried employees	none	Average exp. p. family and p. head (abs. and %) on main items, and in more detail on food, quantities and nutritive values of food for each month of the year. Actual expenditure of each individual family is also given.	Statistische Mitteilungen über den hamburgischen Staat, Nr. 20: Die Lebenshaltung minderbemittelter Familien in Hamburg im Jahre 1925. Hamburg 1926.
1926 (one year)	do.	300 families	—	The report on this enquiry has not been available to the present writer. It was published in Aus Hamburgs Verwaltung und Wirtschaft, vol. 27, No. 5, and vol. 28. Nos. 9, 10, and 11.	

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
1927 (one year)	do.	146 workers	−2500, −3000, −3600, −4300, −5100, >5100, Mk. annual inc. p. fam.	Detailed expenditure p. fam. and p. "man" (abs. and %) on main items, acc. to income p. fam. and number of "men" in the family; food quantities p. "man" acc. to income, separately for each social group. Average prices paid for and nutritive values of food are also given, acc. to income p. "man."	Statistische Mitteilungen über den hamburgischen Staat, Nr. 26: *Die Lebenshaltung der wirtschaftlich schwachen Bevölkerung in Hamburg.* Hamburg 1931.
		102 school teachers	3600, −4300, −5100, −6100, >6100 Mk. annual inc. p. family		
		46 salaried employee	−3600, −4300, −5100, −6100, >6100 Mk. annual income p. family		
1925–1929 (four years)	do.	26 (mainly) workers, identical in each year	none	Average expenditure p. fam. and p. "man" (abs. and %) on main items, and detailed food quantities consumed in each year.	

N.B.: Apart from the above, there exist several other local enquiries which are mentioned in the big report on the enquiry of 1927–28. In addition, several unofficial enquiries have been conducted by workers' organisations, among which a few are mentioned below. For reasons connected with the technique of collective agreements, these reports attach much importance to territorial subdivisions. They contain, however, information (mainly on expenditure, but occasionally also on quantities consumed) which is important in supplementing the great official enquiry of 1927–28. The following may be mentioned:

(1) Zentralverband der Schuhmacher, "300 Haushaltungsrechnungen von Arbeiten der Schuhindustrie und des Schuhmachergewerbes in Deutschland," Nürnberg 1928. Covers 300 shoe workers' families in the year Apr. 1925–March 1926.

(2) Deutschnationaler Handlungsgehilfenverband, "Der Haushalt des Kaufmannsgehilfen," Hamburg, 1927. Covers 290 commercial clerks' families during the year 1926.

(3) Deutscher Baugenwerkabund, "Die Lebenshaltung der Bauarbeiter," Berlin 1931. Covers 896 building workers' families during the year 1929.

(4) Einheitsverband der Eisenbahner Deutschlands, "Die Lebenshaltung des deutschen Reichsbahnpersonals," Berlin 1930. Covers 87 railway workers' and 63 railway employees' families during the year 1929.

(5) Deutscher Landarbeiter-Verband, "Die Lebenshaltung, Lohn- und Arbeitsverhältnisse von 145 deutschen Landarbeiterfamilien," by Wilhelm Bernier, Berlin 1931. Covers 145 agricultural workers' families during the year July 1929–June 1930.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
INDIA May 1921–Apr. 1922 (one month)	Bombay City and Island	2173 workers	−30, −40, −50, −60, −70, −80, −90, >90 Rupees monthly inc. p. fam.	Expenditure on main groups (abs. and %), and detailed exp. on, quantities of, and average prices paid for, food, fuel and light, clothing, miscellaneous, acc. to income for each of the three groups. A large number of sample budgets are given for individual families in the main occupations.	Labour Office, Government of Bombay, *Report on an Enquiry into Working Class Budgets in Bombay*, by G. Findlay Shirras, Bombay 1923. N.B.: contains a detailed bibliography on budget enquiries generally.
		597 workers families consisting of husband, wife and 2 children	same classes.		
		603 single men	−30, −40, −50, −60, −70, −80, >80 Rs. monthly income		
May–Dec. 1925 (one month)	Sholapur City	902 cotton mill workers	−20, −30, −40, −50, −6, −70, −80, −90 Rs. monthly income p. family	Same as for Bombay; but no quantities, no price and no sample budgets are supplied.	Labour Office, Government of Bombay, *Report on an Enquiry into Family Budgets of Cotton Mill Workers in Sholapur City*, Bombay 1928.
Feb.–Aug. 1926 (one month)	Ahmedabad	872 workers	Same as for Sholapur	Same as for Sholapur.	Labour Office, Government of Bombay, *Report on an Enquiry into Working Class Budgets in Ahmedabad*, Bombay 1928.
JAPAN Sept. 1926–Aug. 1927 (one year)	11 large cities 12 industrial centres, 5 mining districts and 6 principal cities 9 prefectures	1575 salaried empl. 3210 wage earners 670 farmers	−60, −80, −100, −120, −140, −160, −180, −200, >200 Yen monthly income p. family	Expenditure (abs. and %) by consumption groups, and for main items in each group, separately for each social class, according to income. No quantities.	XIX° Session de l'Institut International de Statistique, Tokio 1930, "The Family Budget Enquiry in Japan 1926–1927," by J. Matsuda, Tokio 1930. N.B.: The detailed results of this enquiry are available in 4 large volumes (in Japanese only): Kakei Chōsa Hōkōku, Tokyo 1929).
Sept. 1931–Aug. 1932	—	—	−50, −60, −70, −80, −90, −100, >100 Yen monthly income p. family.	Exp. (abs. and %) by consumption groups.	*Résumé statistique de l'Empire du Japon*, 47° année, p. 97, Tokyo, 1933.

NETHERLANDS Sept. 1918–Sept. 1919 (one year)	Amsterdam	82 civil servants	{−1300, −2600, −3400, −4400, −5600, >5600 Fl. annual exp. p. "normal family" of 4 "men".	Detailed exp. (abs. and %) p. family and p. "normal family." No quantities. A special table shows expenditure on food by seasons (quarters).	Statistische Mededeelingen van het Bureau van Statistiek der Gemeente Amsterdam, No. 73: "Les dépenses de 114 ménages de fonctionnaires et d'ouvriers," Amsterdam 1924.
March 1917, March, Sept. and Dec. 1918, March, June, Sept. and Dec. 1919, March, June, Sept. 1920, March 1922.	do.	32 workers	{−8, −12, >12 Fl. weekly expenditure p. "man" in March 1919.	Detailed exp. of each family during each of the 12 months (see col. 1). No quantities.	
Oct. 1923–Sept. 1924 (one year)	do.	212 civil servants and workers	{−1800, −2400, −3600, −5000, −7500, >7500 Fl. annual income p. family.	Detailed exp. (abs. and %) of each family, and for the averages in each income class. Detailed exp. p. "man" in each family, arranged acc. to yearly income p. "man."	Statistische Mededeelingen van het Bureau van Statistiek der Gemeente Amsterdam, No. 80: "Comptes de ménage de 212 familles de différente position sociale," Amsterdam 1927.
Jan.–Dec. 1930 (one year)	Rural section of the commune of Amsterdam	19 non-agriculturalists	{1800, −2400, −3600, −5000, −7500 Fl. yearly income p. family.	Detailed exp. (abs. and %) of each family, and for the average in each income class. A classification is also given acc. to occupation.	Statistische Mededeelingen, as above, No. 96: "Comptes de ménage dans la partie rurale de la commune," Amsterdam 1932.
NORWAY Sept. 1927–Aug. 1928 (one year)	5 large towns	135 workers 31 civil servants	{−900, −1300, −1700, −2100, >2100 Kr. annual exp. p. "man."	Workers: Very detailed exp. p. fam. and "man," still more detailed exp. on food p. "man," with quantities of food p. fam. and p. "man," acc. to exp. p. "man" in each town. Food exp. and quantities p. "man" in each town acc. to number of "men" in the fam. Same information for civil servants, but only for general average and acc. to income for Oslo only. Average prices paid in each town by the workers, and by civil servants in Oslo, acc. to income. Comparison of food quantities with the 1912–13 and 1918–19 enquiries.	Norges Offisielle Statistikk VIII, 103: "Husholdregnskap 1927–1928," Oslo 1929.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical indications and notes (6)
POLAND 1927 (one year)	Warsaw Lodz Dabrowa Basin Upper Silesia	40 workers 32 workers 76 workers 44 workers	−600, −900, −1200, >1200 Zl. annual exp. p. "man"	Detailed expenditure (abs. and %), and quantities for a great number of items of food, and a few fuel and miscellaneous items p. fam. and p. "man" for each income class in each region. Exp. for each individual family is also supplied with the same detail. Nutritive value is given for the average.	Statistique de la Pologne, Tome XL, Fascicule 1: "Budgets des familles ouvrières 1927," Warsaw 1930.
1928 (one year)	Same centres as above, with 23 and 27 and 29 and 14 workers' families, respectively.		Same classes as above	Same information as above.	Statistique de la Pologne, Tome XL, Fascicule 2: Budgets des familles ouvrières 1928, 1929." Warsaw 1933.
1929 (one year)	Same centres, except Upper Silesia, with 18 and 28 and 38 workers' families, respectively.		do.		N.B.: For the Warsaw families, the quantities consumed by each individual family, as well as a few further details are given in: Publications du Bureau de Statistique de la ville de Varsovie, No. 5: "Les Budgets des familles ouvrières à Varsovie 1927–1929," Warsaw 1932. Les résultats de l'enquete sur les "budgets des travailleurs intellectuels, effectuée en mai 1932," by Edward Ostrzeżaki, in Statistique du Travail, published by Office Central de Statistique de la République Polonaise, XIᵉ année 1932, fascicule 4.
May 1932 (one month)	Warsaw	71 intellectual workers	−160, −250, >250 Zl. monthly exp. p. "man."	Expenditure (abs. and %) and quantities acc. to income, for a slightly smaller number of items than above. Nutritive value is also given.	
SOUTH AFRICA 1925 (one month)	9 towns and 5 other areas	442 workers and salaried employees	−£20, −£30, −£41/13/4, monthly income p. family.	Expenditure (abs. and %) p. family on main items, according to income and territorial divisions. Average quantities of food consumed weekly.	Union of South Africa: Report of Cost of Living Committee 1925, Cape Town 1925. N.B.: See also the results of an enquiry conducted in 1932 in Report of the Cost of Living Commission 1932, Pretoria 1932.

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	In some classes distinguished (4)	Information supplied (5)	Bibliographical Indications and notes (6)
SWEDEN about 1923 (one year)	33 towns and 28 industrial centres	747 workers, 445 salaried employees	-825, -1100, -1375, -1650, -1925, >1925, Kr. annual income p. "man"	Detailed expenditure p. "normal family" of 3.3 "men," according to income p. "man," for workers and salaried employees, separately and together, distinguishing towns and industrial centres. Data for middle class shown separately; same according to social class and territorial divisions, and within each division, for each town or centre. Food and fuel quantities p. "normal family," according to social class and income, distinguishing between towns and industrial centres; and according to territorial divisions and individual localities.	Sveriges Officiella Statistik, Socialstatistik: "Levnadskostnaderna i städer och industriorter omkring år 1923." Stockholm 1929.
		208 middle class families	-1100, -1650, -2200, -2750, -3300, >3300, Kr. annual income p. "man"		
1922-23 (one year)	Stockholm	36 workers and 55 salaried employees	-1300, -1950, >1950, Kr. yearly income p. "man"	Detailed exp. (abs. and %) and food quantities p. "normal family" of 3.3 "men," acc. to income for the workers and salaried employees, and on the average for the middle class families. Comparison of the average quantities with those for 1907/08, 1916, 1917, and 1918, as obtained in earlier enquiries.	Stockholm Stads Statistik, X: Special underökningar: "Statistisk Underökning angående levnadkostnaderna i Stockholm." Stockholm 1927.
		27 middle class families	none		
SWITZERLAND 1919 (one year)	Various urban and rural centres	10 higher officials 143 civil servants and salaried empl. 90 skilled and semi-skilled workers 29 unskilled workers	-5000, -6000, -7500, >7500 frs. yearly income p. family	Expenditure is shown p. family, p. "man" and in %, both for main items, and in more detail for foods (particularly meats), in double classifications connecting in various ways any two of the following characteristics: family income, number of "men" in the family, territorial divisions, social class. The same system of classification is adopted for the quantities consumed p. fam. and p. "man."	Eidgenössisches Statistisches Bureau: Schweizerische Statistische Mitteilungen IV. Jahrgang (1922), Heft 1: "Haushaltungsrechnungen schweizerischer Familien aus dem Jahre 1919." Bern 1922.

N.B.: This enquiry was continued, with about the same number of families, up to 1922; but no comprehensive report seems to have been published on the later years. A few indications on the families, and their expenditure acc. to locality and social class (but not according to income within the locality or social class) will be found in: Bundesamt für Industrie, Gewerbe und Arbeit: "Ergebnisse der schweizerischen Sozialstatistik, abgeschlossen auf Ende 1931," Bern 1931, pp. 56-62.

Year	Place	Number of families	Income groups	Information given	Source
1919 (one year)	Zürich	51 civil servants and employees, 34 workers	none	Expenditure (abs. and %) p. fam. and p. "man" on main items, and in more detail on foods and fuels; quantities (p. fam. and p. head) and prices of foods and fuels are also given. The classification is according to occupations only (8 occupational groups).	Statistisches Amt der Stadt Zürich: Statistik der Stadt Zürich, Heft 28: "Zürcher Haushaltsrechnungen aus dem Jahre 1919," Zürich 1919.
1912 (one year)	Winterthur and various rural communes of the canton of Zürich	130 (mainly) workers	-2000, -2500, -3000, -3500, -4000, >4000, frs. annual income p. family	Detailed expenditure (abs. and %) p. family, and quantities of food p. fam. and p. "man," according to occupations (10 groups), territorial divisions (7 districts), and income p. family.	Statistische Mitteilungen betreffend den Kanton Zürich, Heft 141: "Haushaltsrechnungen aus der Stadt Winterthur und den Landgemeinden des Kantons, betreffend die Jahre 1912, 1919 und 1920." Winterthur 1922.
1919 (one year)	do.	37	-4000, -5000, -6000, -7000, >7000 frs. annual inc. p. family	Same information as above, but exp. is also given p. "man" and acc. to number of "men" in the family.	Same series, Heft 150: "Haushaltsrechnungen (as above) betreffend die Jahre 1921 und 1922," Winterthur 1925.
1920 (one year)	do.	53	do.		
1921 (one year)	do.	44	-4000, -5000, -6000, -7000, -9000, >9000 frs. annual income p. family	Same information as for 1919 and 1920	
1922 (one year)	do.	56	do.		
1912 (one year)	Basel	32 sal. empl. and 46 workers	-2500, -3300, -4500, >4500 frs. annual income p. family; -1000, -1250, -1500, >1500 frs. annual income p. "man"	Expenditure on main items, and in more detail on foods, p. family, p. "man" and in %, according to social class, number of "men" in the family, income p. family, and income p. "man." Quantities of foodstuffs are given p. family and p. "quet" according to the same criteria.	Mitteilungen des Statistischen Amtes des Kantons Basel-Stadt, No. 45: "Haushaltsrechnungen von Basler Familien aus den Jahren 1912, 1919-1923." Basel 1925.
1919 (one year)	do.	16 sal. empl. and 21 workers	-5000, -6000, -7500, >7500 frs. annual income p. family; -2000, -2500, -3000, >3000 frs. annual income p. "man"		
1920 (one year)	do.	10 sal. empl. and 14 workers			
1921 (one year)	do.	28 sal. empl. and 36 workers			
1922 (one year)	do.	26 sal. empl. and 24 workers			
1923 (one year)	do.	15 sal. empl. and 18 workers			

Country, period and duration (1)	Localities covered (2)	Number and kind of families (3)	Income classes distinguished (4)	Information supplied (5)	Bibliographical Indications and notes (6)
1912, 1918–1923 (seven years)	Bern	Workers and salaried employees, varying from 11 to 67 in number	none	Average expenditure p. family, p. head, p. "man," and in %, in each year on main items, and in more detail on food, clothing, housing and heating. Average quantities of foodstuffs p. family, p. head and p. "man" in each year.	Statistisches Handbuch der Stadt Bern, herausgegeben vom Statistischen Amt, Erste Ausgabe 1925, pp. 204-226.
UNITED KINGDOM June 1918 (one week)	Great Britain	566 skilled workers 139 semi-skilled w. 266 unskilled w. 104 clerks 231 "on service"	none	Detailed expenditure on, and quantities of, food p. family of 4.57 "men" according to occupational groups. Total expenditure on food p. "man" by occupations and districts, *with standard deviations*. Comparisons with the 1904 Board of Trade enquiry. Detailed expenditure on other budget items, except clothing, with quantities of fuels used.	Report of the working classes Cost of Living Committee 1918, London 1918, Col. 8980.

UNITED STATES OF AMERICA

Consult for the very numerous enquiries conducted in the U.S.A.: Faith M. Williams (assisted by Helen Connolly), Bibliography on Studies of Costs and Standards of Living in the United States. Preliminary Report, July 1930, with a Supplementary Bibliography, January 1932. U. S. Department of Agriculture, Bureau of Home Economics (Stereotyped).

General References.

Bulletins of the Russell Sage Foundation Library, entitled "Family Budgets and Costs and Standards of Living." The last one issued is Bulletin No. 120, August 1933. Library of Congress, Select List of References on the Cost of Living and Prices, compiled under the direction of Hermann Henry Bernard Meyer, Washington, Govt. Printing Office, 1910.

"Recent Family Budget Enquiries," *International Labour Review*, vol. XXVIII, No. 5, Nov. 1933, pp. 635-672.

Resolutions of the IIIrd International Conference of Labour Statisticians, in *International Labour Review*, vol. XV, No. 1, Jan. 1927, pp. 17-19; also given in "The Third International Conference of Labour Statisticians," International Labour Office, Studies and Reports, Series N, No. 12.

The enquiries conducted by the British Board of Trade reference to which has been made above are the following:

Cost of Living of the Working Classes. Report of an Enquiry by the Board of Trade . . . in the principal industrial towns of the United Kingdom, London 1908, Cd. 3864.

Cost of Living in German Towns. Report of an Enquiry etc. . . . in the principal industrial towns of the German Empire. London 1908, Cd. 4032.

Cost of Living in French Towns. Report etc. . . . in the principal industrial towns of France. London 1909, Cd. 4512.

Cost of Living in Belgian Towns. Report etc. . . . in the principal industrial towns of Belgium. London 1910, Cd. 5065.

Cost of Living in American Towns. Report etc. . . . in the principal industrial towns of the United States of America, London 1911, Cd. 5609.

Cost of Living of the Working Classes. Report etc. . . . in industrial towns of the United Kingdom. London 1913, Cd. 6955.

LÉON WALRAS ET SA CORRÉSPONDANCE AVEC AUGUS-TIN COURNOT ET STANLEY JEVONS

AVEC UNE NOTE D'ÉTIENNE ANTONELLI[1]

LA CORRÉSPONDANCE de Léon Walras constitue la mine de documents la plus riche pour quiconque veut aller aux sources de toutes les doctrines modernes d'économie pure.

Pour montrer, sur un exemple, l'intérêt scientifique qu'aurait la publication de cette correspondance inédite que l'Econometric Society a bien voulu patronner dans les conditions qui seront par ailleurs publiées dans cette revue, j'ai choisi quelques extraits dont l'intérêt historique est évident.

On sait que Léon Walras fut, en France, le premier économiste qui ait appliqué le calcul mathématique au problème de l'équilibre économique général.

Mais avant lui, deux techniciens avaient appliqué le même langage à la résolution de problèmes particuliers d'économie politique: ce sont Cournot et Dupuit.

Le premier avait publié en 1838 un ouvrage intitulé: *Recherches sur les principes mathématiques de la théorie des richesses* qui passa à peu près complètement inaperçu des savants de son époque, mais qui fut brusquement tiré de l'oubli en 1873 quand Walras, dans un mémoire qu'il présenta à l'Académie des sciences morales et politiques reconnut Cournot comme son maître.

Le second avait publié dans les *Annales des Ponts et Chaussées* en 1844 et 1849 deux articles, l'un sur "La mesure de l'utilité des travaux publics," l'autre sur "L'influence des péages sur l'utilité des voies de communication" où, pour la première fois, était analysée la notion de "gradation" de l'utilité.

Or, Walras eut l'occasion dans une lettre adressée directement à M. Cournot et dans une autre adressée à W. S. Jevons de marquer le rôle qu'avaient joué ces deux précurseurs dans l'élaboration de ses travaux.

Ce sont ces documents qui sont ci-dessous publiés:

I. LETTRE DE A. COURNOT À LÉON WALRAS

Villexon, 3 Septembre 1873

Monsieur,

J'ai reçu votre lettre de Lausanne et les feuilles que vous avez eu la bonté d'y joindre, dont je suis fort reconnaissant. Je les ai lues avec

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toute l'application que je pouvais mettre à cette lecture; car je dois vous dire que depuis 30 ans je suis obligé de recourir à un lecteur pour ma pâture quotidienne; bien entendu que je ne trouve pas de garçon capable de me lire des mathématiques, ou que je ne peux pas lire des mathématiques avec les oreilles, encore moins en dicter comme feu Léonard Euler, et c'est ce qui m'a obligé depuis 30 ans à renoncer aux mathématiques. Enfin je vous ai lu avec mes yeux, le mieux ou le moins mal que j'ai pu, et naturellement en saisissant avec plus de facilité en vertu de mes anciens souvenirs, les assemblances que les dissemblances. Que l'on aille comme vous du général au particulier ou, comme j'avais essayé de le faire, du cas particulier au cas général, c'est une question de méthode. La mienne me semble plus rapide, la vôtre marche à pas plus comptés, ce qui donne lieu de croire qu'elle marche à pas plus sûrs.

On jugerait mieux de la question du fond si l'on avait sous les yeux ce qui précède et surtout ce qui doit suivre.

Pour arriver à la question vraiment intéressante, celle du libre échange ou du libre troc international, j'ai été obligé de recourir à un nouveau *postulat*, celui de la *compensation* des effets secondaires ou dérivés; et une fois ce postulat admis (lequel n'a rien de spécial à la question actuelle et se retrouve à peu près, sous une forme ou une autre, dans toutes les applications des mathématiques) il me semble que les exemples arithmétiques peuvent dispenser de recourir aux signes de l'algèbre.

C'est ce que j'ai essayé de faire dans mon volume de 1863, qui a eu encore moins de succès chez tous les économistes que ma *plaquette* de 1838. Si votre méthode vous dispense de recourir à ce postulat ou à quelque postulat équivalent, ce sera un progrès incontestable auquel j'applaudirai plus vivement que personne, car je me rendrai compte de la difficulté vaincue. Mais je tremble que vos courbes "d'utilité intensive et extensive" ne vous mènent au pur laissez-faire, c'est à dire en économie nationale au déboisement du sol et en économie internationale à l'étouffement de la plèbe des races par une race privilégiée conformément à la théorie de M. Darwin.

Mon fils ira vers la fin de ce mois à Delémont et je me conformerai à mon rôle de grand-père: je garderai les petits-enfants qui ont perdu leur mère et leur grand-mère. Je pense que nous retournerons à Paris vers la fin d'octobre. Notre adresse parisienne est Carrefour d'Odéon, N° 10 et il ne faut pas confondre le carrefour avec la rue ni avec la place de l'Odéon.

Comme on vient toujours à Paris, sauf les cas de bombardement ou de pétrole, j'espère bien avoir le plaisir de vous y voir dans les huit mois que nous y passons d'ordinaire; et, en attendant, je vous renouvelle l'expression de ma gratitude et de ma haute considération.

<div style="text-align: right">COURNOT</div>

II. LETTRE DE LÉON WALRAS À A. COURNOT

<div style="text-align: right">Lausanne, 20 Mars 1874</div>

Monsieur,

J'espèrais pouvoir vous envoyer beaucoup plus tôt mon Mémoire de l'Institut. Il n'a paru qu'en février dans le N° de janvier du *Bulletin* de l'Académie des Sciences morales et politiques et j'en ai reçu ces jours-ci seulement, le tirage à part. Je m'empresse de vous en adresser un exemplaire. J'attendais cette occasion pour répondre à votre dernière lettre; je vais essayer de le faire de mon mieux. Mais ne vous astreignez pas, je vous en supplie, à relever mes observations: c'est une explication que je vous fournis et non pas, comme vous le verrez, une discussion que j'engage.

Plus je me mets en possession de ma propre idée, plus je me convaincs, Monsieur, qu'il n'y a pas précisément entre nos recherches une différence de méthodes, plus ou moins rapides l'une que l'autre mais seulement une différence de positions. Vous avez attaqué le problème sur un point et moi sur un autre; nos deux tranches sont différentes mais voisines; c'est ce que j'espère vous montrer aujourd'hui.

J'achève en ce moment la Théorie de l'Échange de plusieurs marchandises entre elles. Il y a dans ce cas un double èquilibre à étudier: 1°, l'équilibre des prix des marchandises, deux à deux et 2°, l'équilibre général qui a lieu lorsque le *prix de deux marchandises quelconques l'une en l'autre est égal au rapport du prix de l'une et l'autre en une troisième quelconque.* Ce sont les arbitrages qui amènent l'équilibre général, à supposer que le premier équilibre existe seul. A cet état d'équilibre général qui est l'état d'équilibre définitif, on voit clairement la *valeur d'échange* apparaitre comme un des termes indéterminés et arbitraires desquels seulement la proportion est la même que la proportion commune et identique des *raretés*, c'est à dire *intensités des derniers besoins satisfaits* chez tous les échangeurs et desquels, par conséquent, les rapports seuls, qui sont les *prix égaux* aux rapports des raretés de deux marchandises quelconques chez un porteur quelconque, sont susceptibles de recevoir une valeur numérique déterminée.

(A), (B), (C), (D) étant les marchandises, V_a, V_b, V_c, V_d la valeur d'échange, r_{a1}, r_{b1}, r_{c1}, $\cdots$ r_{a2}, r_{b2}, r_{c2}, $\cdots$ r_{a3}, r_{b3}, r_{c3}, r_{d3}, $\cdots$ les raretés chez les échangeurs (1), (2), (3) cette circonstance capitale s'exprime avec une parfaite simplicité de la manière suivante:

$$V_a : V_b : V_c : V_d \cdots$$
$$:: r_{a1} : r_{b1} : r_{c1} : r_{d1} \cdots$$
$$:: r_{a2} : r_{b2} : r_{c2} : r_{d2} \cdots$$
$$:: r_{a3} : r_{b3} : r_{c3} : r_{d3} \cdots$$
$$:: \cdots \cdots \cdots$$

Il y a deux restrictions à faire: l'une relative à la non-demande d'une marchandise par un échangeur, auquel cas, le terme de rareté relatif à cette marchandise manque dans la proportion relative à cet échangeur; la seconde, relative à la discontinuité des courbes de rareté, au quel cas la proportionalité des valeurs d'échange et des raretés n'est plus rigoureuse.

Là, Monsieur, est, je crois, le point essentiel à saisir, dans l'analyse du fait de la valeur d'échange. Je vais m'efforcer de vous en faire voir l'importance.

Vous avez posé plus hardiment que personne la question des variations absolues de valeur et vous l'avez tranchée dans un sens contraire à ce que vous avez appelé le *préjugé* des économistes. Vous avez très bien aperçu qu'au delà des changements dans les rapports de valeur ou dans les prix il y avait à considérer les changements dans les conditions absolues de la valeur, c'est à dire dans l'utilité et la quantité des marchandises, grâce à l'artifice qui fait rentrer l'utilité elle-même dans les calculs. Il me semble que, dans ma théorie, la question se présente avec encore plus de clarté et de précision. Les rapports V_a/V_b par exemple, ayant changé, les rapports $\dfrac{r_{a1}}{r_{b1}} = \dfrac{r_{a2}}{r_{b2}} = \dfrac{r_{a3}}{r_{b3}} = \cdots$ ont changé. Or, ces derniers rapports sont les seuls dont nous ayons le droit de considérer les deux termes comme ayant chacun une valeur déterminée et absolue. Donc, c'est seulement à leur sujet qu'on peut se demander ceci: sont-ce les raretés ou les intensités des derniers besoins satisfaits de (A) ou bien sont-ce les raretés ou les intensités des derniers besoins satisfaits de (B) (C) (D) $\cdots$ qui ont changé chez les échangeurs. Or, ici, il faut distinguer.

Théoriquement, il est impossible que les raretés d'une marchandise aient changé sans que les raretés de toutes les marchandises aient changé. Je suppose que l'utilité de (A) par exemple, ait augmenté ou que sa quantité ait diminué, toutes les raretés de (A) auront changé; mais toutes les raretés de (B) (C) (D) $\cdots$ auront ausst changé par cela même. Je ne vous fais pas ici la démonstration de cette proposition; mais je puis vous assurer que vous la trouverez établie d'une manière incontestable dans ma théorie de l'échange de plusieurs marchandises entre elles, à supposer que, dès à présent, vous ne l'aperceviez pas de vous-même. Ainsi au point de vue théorique, les économistes (et je le ferai moi-même en leur nom) pourraient vous objecter que les mobiles représentant les marchandises que vous avez si ingénieusement supposées en mouvement sur une ligne droite et dont les distances variables correspondent aux prix variable des marchandises les unes avec les autres sont tout justement liés entre eux par une relation telle que le changement de position de l'un entraine le changement de position de tous les autres; et que, par conséquent, on ne peut concevoir théoriquement un changement de prix provenant d'un changement d'un seul des deux termes de valeur et non de l'autre.

Mais cela dit, et *pratiquement,* il est bien certain que, dans l'exemple cité, les changements dans les raretés de (B) (C) (D) $\cdots$ dont l'utilité et la quantité n'ont pas changé sont insignifiants et que les seuls changements dans les raretés de (A) dont l'utilité a augmenté ou dont la quantité a diminué, sont sensibles *si les marchandises en présence sont en grand nombre et en quantité considérable.* Par conséquent, vous avez, en un sens, raison contre les économistes lorsqu'en vertu d'une application fondée et légitime de la *loi des grands nombres,* vous considérez que c'est la valeur de (A) qui a augmenté, les valeurs de (B) (C) (D) $\cdots$ étant restées les mêmes. Et vous avez aussi raison contre eux dans les conséquences importantes que vous tirez de cette manière de voir.

A ce premier exemple, j'en pourrai joindre un second tiré de vos courbes de débit. Théoriquement, pour que le prix déterminé au moyen

de cette courbe fût définitif, il faudrait une condition qui ne sera presque jamais remplie d'une manière absolue: savoir que le détournement des demandes et offres effectives des autres marchandises se fît proportionnellement et de façon à ne pas troubler l'équilibre des prix de ces marchandises. Et toutefois, pratiquement ici aussi, on peut faire abstraction de cette condition.

Pardonnez-moi, Monsieur, ce long développement; il était nécessaire pour vous faire reconnaître, comme je le reconnais moi-même, de plus en plus, quelle est notre situation respective?

Notre méthode est la même, car la mienne est la vôtre, seulement, vous vous placez immédiatement au bénéfice de la loi des grands nombres et sur le chemin qui mène aux application numériques. Et moi je demeure en de ça de cette loi sur le terrain des données rigoureuses et de la pure théorie. Mais mon analyse terminée, je vous rejoins; et à partir de ce moment, je n'aurai plus qu'à vous suivre. Je m'en suis assuré en matière d'échange, de valeur d'échange il me reste à m'en assurer en matière de production, de frais de production, d'impôt etc. Cela dit, me reprocherez-vous ma lenteur? La phrase dans laquelle vous m'en parlez peut passer pour un éloge et c'est ainsi que je l'ai prise. Et en effet, la paresse de notre temps à scruter longuement et patiemment les principes économiques et sociaux n'est-elle pas la raison pour laquelle toutes les questions économiques et sociales, résolues tous les matins par nos journalistes, se reposent tous les soirs exactement dans les mêmes termes que la veille? Mon père qui n'était pas spécialement mathématicien, quoi qu'il sût bien la géométrie, mais qui était excellent philosophe, avait commencé, dès 1831, à approfondir avec les ressources de la logique ordinaire, les principes fondamentaux de l'économie politique. Vous avez fait, en 1838 avec les ressources du calcul un effort dans ce sens. Et moi, je me crois en bonne voie, quand, marchant sur vos traces et profitant de vos travaux à tous deux, je tache de creuser encore un peu plus avant pour trouver le sol résistant sur lequel seul nous pourrons édifier la science sociale. Quant aux conséquences lointaines "du pur laissez-faire" que vous entrevoyez au delà de mes premisses faites-moi crédit, Monsieur, d'un peu de temps et de confiance et vous verrez que je saurai les éviter. L'économie politique pure n'est que la théorie de la richesse sociale. Pour passer de là à l'économie politique appliquée et à l'économie sociale, il faut introduire en scène l'homme physiologique et psychologique, individuel et moral. Croyez que c'est un élément dont je saurai tenir bon compte. Maintenant, et si votre patience m'a suivi jusqu'ici, permettez-moi de terminer cette lettre par quelques considérations plus personnelles et plus pratiques.

Vous avez attendu 35 ans, Monsieur, que quelqu'un vous suivit dans cette voie d'application des mathématiques à l'économie politique et sociale, qui sera, peut-être, une des plus fécondes d'entre les applications des mathématiques aux sciences physiques qu'aura engendrée la conception de Descartes sur la généralité des rapports entre grandeurs. Me faudra-t-il faire preuve de la même patience? Je vous assure que je ne m'y résigne pas encore et que j'ambitionne d'avoir et de vous donner à vous-même la plaisir de voir d'ici à quelques années mes idées prendre droit de cité dans la science. C'est

pourquoi, non content de creuser ma théorie, je ne néglige aucun
moyen de la produire. Le plus convenable et le plus sûr, m'a paru
être la composition d'un *traité d' économie politique et social* rédigé
d'après cette méthode et qui répond à l'importance du développement
des phénomènes économiques qui est effectivement de notre siècle.
Nos anciens traités qui n'ont guère été renouvelés sont un peu l'équival-
ent en économie politique de ce que serait celui de l'abbé Hollet en
physique. Je crois avoir les matériaux d'un ouvrage plus digne de notre
époque, et j'en ai entrepris d'abord une première esquisse en trois
volumes qui n'est autre chose que l'ensemble de mon cours reparti
en trois années aux étudiants de l'Académie de Lausanne. J'ai ici un
imprimeur qui s'est engagé avec moi pour un demi-volume. Si cette
première partie s'écoule, il entreprendra successivement les cinq
autres. Mes éditeurs, à Paris, sont MM. Guillaumin et Cie qui reçoi-
vent le dépôt de mes exemplaires avec 50% net de remise; j'aurais le
désir de pouvoir leur associer MM. Hachette et Cie aux mêmes condi-
tions. Serait-il indiscret de vous demander de vouloir bien leur en
faire la proposition? puisque vous avez eu la bonté de faire allusion ua
souvenir de mon père, je prends la liberté de l'invoquer ici en vous
adressant ma prière. Il était, comme vous, le camarade de feu M. L.
Hachette à l'École Normale. Je ne sais si les chefs actuels de la puis-
sante maison qu'il a fondée seraient touchés de cette circonstance;
mais peut-être verront-ils une affaire sérieuse dans la publication
d'un traité élémentaire d'économie politique établi sur un plan nou-
veau, s'inspirant d'une méthode originale tenu au courant du progrès
des faits et susceptible de grands développements ultérieurs. Dans
les conditions où je la présente, cette affaire ne leur ferait courir aucun
risque et perte d'argent. Ils y entreraient plus ou moins, ensuite, selon
la tournure qu'il lui verraient prendre. Par conséquent, tout ce que
j'oserais solliciter de vous serait de leur assurer d'après ce que vous en
pourrez savoir, qu'elle ne les exposerait pas non plus à aucune chance
de déconsidération par suite d'une trop grande médiocrité de l'auteur
et de son œuvre. Pour le cas où vous croiriez pouvoir vous charger de
cette demande je joins à la présente une note explicative. Et que si,
au contraire, ma demande vous paraissait déplacée, tenez-la, tout sim-
plement, pour non avenue.

Veuillez agréer, Monsieur, avec mes excuses et mes remerciements
à l'avance; et quoi qu'il arrive, l'expression de mes sentiments les plus
respectueux et les plus dévoués.

<div align="right">LÉON WALRAS</div>

III. LETTRE DE S. JEVONS À LÉON WALRAS

<div align="right">28 Février 1877</div>

Monsieur,

Je vous remercie mille fois de m'avoir envoyé des exemplaires de
vos quatre mémoires qui me sont bien parvenus. Je suis heureux de
voir que vous continuez vos recherches et que maintenant vous avez
créé une théorie de la Capitalisation. Dans ces mémoires se revèlent
l'originalité et le talent qui ont fait de vos précédents écrits des ouv-
rages si remarquables.

Pour l'instant je ne m'occupe pas beaucoup d'Économie politique, la majeure partie de mon temps est employée à reviser mon livre intitulé "Principes de science" pour le faire paraitre en seconde édition, la première étant épuisée. Ce travail ne me permet pas de me livrer à d'autres lectures.

Cette opération achevée, je me propose de complèter mon étude sur la philosophie de John Stuart Mill et je montrerai que la valeur logique de ses écrits est loin d'être appréciée exactement, que Stuart Mill est en réalité un mauvais logicien.

J'ai le plaisir de vous envoyer par la poste un exemplaire d'un traité élémentaire de logique à l'usage des écoles. Les livres de ce genre se vendent en grande quantité en Angleterre et en Amérique. J'ai aussi l'intention de préparer un traité élémentaire d'Économie politique pour Messrs. Macmillan.

Ma conférence inaugurale parue dans la *Fortnightly Review* va bientot être reproduite, traduite en français par Mr. de Fortfertins, dans le *Journal des Économistes,* numéro de Mars; c'est du moins ce dont on m'a informé.

Il y a quelques semaines, j'eus l'occasion de lire le "Mémoire de la mesure de l'utilité des Travaux publics" par Dupuit, *Annales des Ponts et Chaussées,* 1844, mémoire que je ne connaissais pas encore. Il est impossible de ne pas admettre que Dupuit ait eu une parfaite compréhension du sujet et nous précéda dans la question des idées fondamentales de l'utilité; néanmoins il ne traita pas le sujet complètement, il n'arriva pas à la théorie de l'échange. Il est extraordinaire de voir le peu de bruit que sa publication fit dans le monde des économistes; la plupart ignorent son existence.

J'espère que votre santé est rétablie, et qu'il vous est possible de vous épargner les excès de travail.

Je suis maintenant installé confortablement à Hampstead; mes conférences à University College et quelques autres travaux scientifiques me laissent le temps de me reposer, ou, si bon me semble, de me livrer à des travaux littéraires.

Si jamais vous venez à Londres, je serais très heureux de recevoir votre visite, et si vous n'avez pas encore vu l'Angleterre, cela vous serait une occasion de visiter les lieux remarquables de la contrée. Je parle le français *très mal,* peut-être parlez-vous anglais? En tout cas nous ferions en sorte de rendre possible la discussion de matières nous interessant l'un et l'autre. Cependant il faut que je vous prévienne qu'à la fin de juin je partirai pour la Norvège d'où je ne reviendrai que vers la fin d'août. Croyez-moi, etc.,

<div align="right">W. S. JEVONS</div>

IV. LETTRE DE LÉON WALRAS À M. STANLEY JEVONS

<div align="right">Chateau de Glérolles, 25 Mai 1877</div>

Cher Monsieur,

J'ai été empêché de répondre plus tôt à votre dernière et aimable lettre par plusieurs circonstances de famille ou d'affaires et principalement par l'obligation que je me suis imposée de pousser l'impression de la seconde partie de mes *Éléments d'économie politique pure* de façon

à ce qu'elle soit terminée cet été. Mais je ne veux pas tarder plus long-temps à vous remercier de vos communications ainsi que de l'envoi de votre petit traité de logique. Je l'ai lu et l'ai passé ensuite à mon éminent collègue, le professeur Charles Secrétan. Comme son autorité est considérable en ces matières, je suis heureux de pouvoir vous dire qu'il en a été aussi satisfait que moi, qu'il a beaucoup loué votre ex-position de la méthode inductive, et trouvé dans votre jugement sur Bain la marque d'un esprit vigoureux et indépendant. Sur quoi je lui ai assuré que c'était bien tout justement ce que vous étiez en éco-nomie politique comme en philosophie.

Je n'hésite pas du reste à croire que la valeur de votre économie politique tient en grande partie à la supériorité de votre méthode philosophique; et je ne fais pas non plus difficulté de considérer avec vous J. S. Mill comme aussi pauvre logicien que médiocre économiste, malgré la peine incroyable qu'il se donne pour ne pas réussir à faire ses démonstrations. Par exemple, je dois vous avouer que je ne suis pas de votre avis sur le mérite des Mémoires de Monsieur Dupuit pub-liés en 1844 et 1849 dans les *Annales des Ponts et Chaussées*.

En ce qui touche la détermination du prix dans le cas de monopole, M. Dupuit n'a fait que reproduire la théorie de M. Cournot. Les seules choses qui lui appartiennent en propre sont les observations relatives à la multiplicité possible des prix dans le même cas, et son expression mathématique d'utilité, mais celle-ci est radicalement inexacte. Cette théorie consiste à voir la mesure de l'utilité dans le sacrifice pécuniaire maximum que les consommateurs sont disposés à faire pour se procurer un produit, c'est à dire dans l'aire de la courbe de demande. Or, sans doute, ce sacrifice pécuniaire dépend en partie de l'utilité du produit; mais il dépend aussi, en partie, de l'utilité des autres produits; et il dépend aussi, en partie, de la quantité de richesse, évaluée en monnaie que possède le consommateur. En termes précis, l'aire de la courbe de demande est *fonction non* seulement de l'utilité du produit à demander (exprimée par une courbe d'utilité) mais aussi de l'utilité de tous les autres produits qui sont sur le marché (exprimé de la même manière) et aussi enfin des moyens du consommateur. En dernière analyse la théorie de M. Dupuit me parait consister dans une confusion complète de la courbe d'utilité et de la courbe de demande. C'est M. Cournot qui a trouvé celle-ci; c'est vous qui avez trouvé la première et c'est moi qui ai trouvé comment il fallait tirer l'une de l'autre. Quant à M. Dupuit, je soutiens qu'il n'a rien à réclamer ici. En revanche je vous indiquerai un bon article de lui sur l'objet et le caractère de l'économie politique pure, dans le *Journal des Économistes* de juillet 1861.

Puisque je fais tant que d'aborder les discussions scientifiques, il faut que je vous dise que j'ai récemment étudié à fond votre théorie de la rente à propos de la réfutation que je fais dans mon cours et que je publie dans mes *Éléments*, de celle de Ricardo. Le vice principal de la théorie de Ricardo, comme aussi de celle de MacCulloch telle que vous la reproduisez résumée en quatre points, c'est de reposer sur le principe que "le produit de la terre ne peut être indéfiniment accru en proportion des *frais*"(*outlay*). En effet, le principe ainsi énoncé ne serait vrai que si des *frais* proportionnels correspondaient

à des *quantités* proportionnelles de services des capitaux et de services personnels, c'est à dire que si la valeur de ces services était constante; car autrement, et si on admet que la valeur en question diminue, rien n'empêche qu'on puisse avoir un accroissement de produits proportionnels à l'accroissement des frais. Or, il est faux que la valeur des services des capitaux et services personnels soit constante et qu'il n'y ait en conséquence de rente que par l'effet d'une charté croissante des produits agricoles. Il y a rente sans qu'il y ait enchérissement de cette sorte.

Vous évitez cette erreur en parlant, quant à vous, de doses et d'incréments, de capital et de travail, doses et incréments mesurés non par leur prix, mais par leur quantité! Et ainsi le principe fondamental devient vrai. Mais alors, reste à savoir s'il est susceptible d'être exprimé dans la forme mathématique que vous lui donnez, c'est à dire dans la forme d'une fonction décroissante d'une seule variable. J'avoue que je ne le crois pas, parce que je ne crois pas que les services des capitaux soient réductibles en travail, et parce que, le fussent-ils, il ne le seraient pas en une seule espèce de travail. Il faut admettre nettement que les produits agricoles ou industriels, résultent de la combinaison d'espèces multiples de services personnels, de services des capitaux et de services des terres et chercher une forme mathématique qui soit assez large pour se prêter à l'expression de cette multiplicité dans le problème de la détermination des prix des produits et des services producteurs. C'est à quoi il me semble que seuls mes *coefficients de production* réussissent.

J'aurais le plus vif désir de discuter avec vous ce point capital, et après y avoir bien réfléchi, je suis vivement tenté d'accepter l'offre séduisante que vous me faites, en allant moi-même vous porter mon ouvrage à la fin des vacances. Vous me dites que vous serez absent en juillet et août. Or, c'est justement au commencement de septembre que j'espère avoir mon demi-volume imprimé; et j'irai, fort probablement, à cette époque, à Paris, pour le recommander à quelques revues et journaux, en même temps que je ferai à Chartres une visite à ma mère. Ce serait une occasion de pousser jusqu'à Londres et d'y faire votre connaissance. Mr. Secrétan me félicitait l'autre jour bien vivement de m'être rencontré scientifiquement avec un homme tel que vous, et je puis vous dire en toute sincérité que je m'applaudis tous les jours de cette rencontre: J'ai la conscience qu'elle m'a donné tout d'abord une parfaite sérénité d'âme, quant à la valeur de mes travaux; j'ai de plus, la persuasion qu'elle a singulièrement hâté le résultat de mes efforts; je suis enfin convaincu qu'une entente encore plus complète avec vous me serait toujours aussi avantageuse sous tous ces rapports. Et, puisque, dans la haute situation que vous occupez, vous voulez bien me traiter en ami, je serai ravi, si rien ne vient m'en empêcher, d'accepter votre gracieuse invitation avec autant de simplicité et d'empressement, que vous me l'avez faite.

Recevez, cher Monsieur, l'expression de mes sentiments bien dévoués.

<div align="right">Léon Walras</div>

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STABLE PRICES VS. STABLE EXCHANGES

By C. Reinold Noyes

Since the war, we have heard many proposals to abandon fixed exchange rates for a national money in order to enable us to aim instead at stability in its internal purchasing power, free from external interference. Two assumptions are tacit in such proposals. The first is that the two objectives are incompatible with each other. Experiences since the war have seemed to support this assumption—at least under the conditions that have prevailed and still prevail. The second—in part a corollary—is that in order to be free to pursue one aim all that is necessary is to sacrifice the other. Since the second assumption, particularly, forms the theoretical basis of much of the economic nationalism now prevalent, it is important to determine whether or not it is sound. Setting this limited objective for a study of the question, it is advisable entirely to disregard the allied questions as to which of the two objectives is to be preferred for itself and as to whether stability in price levels is in itself desirable.

It is generally agreed that the pre-war gold standard aimed almost exclusively at stable exchange rates. It worked to keep the price levels of the countries using this standard in rough conformity to each other but to subject each country to sufficiently influential movements in price levels elsewhere. It relied on the comparative inflexibility in the total quantity of world monetary gold to maintain comparative stability in all price levels. History shows that in this respect it was ineffective. True, it hampered or isolated local inflations, but it did not prevent or isolate local deflations. Local inflation was rendered in large part non-contagious, so long as the gold standard was generally maintained, but local deflation was definitely contagious.

Obviously, the proposal to aim at stable internal prices and, for that purpose, to permit the exchange to fluctuate responsively involves the abandonment, for the time being, of the old gold standard and remaining off gold until at least the greater part of the world adopts and attains the same objective, and thereafter until exchange rates come into adjustment with the then existing relations between the various price levels. Therefore, in examining the assumption that it is necessary to sacrifice stability of exchanges in order to secure stability of prices, it is essential, for the purpose of analysis, to eliminate gold movements both as a factor in balancing international payments and as an automatic control of the volume of circulating media and thereby of prices. This does not mean, however, that historical studies of former régimes of inconvertible paper money are wholly applicable. There is a real

difference between 20th century conditions under programs of managed money and 19th century conditions under inconvertible paper, in that the former seek deliberately to control the situation, while the latter usually arose from lack of control.

There are two general mechanisms by which it is proposed to control and stabilize the internal price level. The first is to manage the fluctuations in the exchange rates by means of an "equalization" fund. In a country like England, where most materials are imported and much of the product is exported, this may accomplish a good deal, since the internal price level must to that extent conform to foreign prices reflected through the manipulated exchange rates. The stability of English prices since early 1932 indicates that this mechanism has worked fairly well under the then existing conditions. Now the United States has established a similar fund, ostensibly for a similar purpose. But stabilization or equalization funds operating in a world off gold would need to speculate in inconvertible foreign monies. It is hardly conceivable that they would be willing to do so. The British plan has worked as long as other countries have maintained a free and fixed market for gold. It would probably not be attempted in case of complete abandonment of gold redemption or of free gold markets.

The second mechanism is the management of the fluctuations in the quantity of total circulating media within the country. This is based on the power to contract or expand the reserves by means of open market operations or through encouraging or discouraging rediscounts by means of changes in basic interest rates. In turn, this mechanism depends for its effectiveness on:

1. Whether the complex of interest rates rises and falls with the rediscount rate and, as a result, induces changes in the quantity of total media.

2. Whether the quantity of total media increases and decreases with the increases and decreases of reserves provided by open market operations and/or rediscounts.

3. Whether prices rise and fall, or stop falling and rising, with increases and decreases of total media (i.e., the other two terms in the equation of exchange maintain a constant relation between themselves).

All of these three assumptions are open to question. Experience in the United States during the past decade indicates that changes in volume of total media are a natural (social) and not a mechanical process. Contraction can be compelled. Expansion cannot. Recently the mechanism has not been working well. Nevertheless, in order to confine this study to the validity of the assumption that there is a free choice between the objectives of stable prices and of stable exchanges,

it is necessary to eliminate consideration of this question and to adopt
the hypothesis that the objective is attainable. For the purpose of this
analysis we will take it for granted that management of money can
secure stability of price levels if it is not defeated by hitherto uncon-
sidered interfering factors. This, being interpreted, means that, if
internal prices are declining from the chosen level, interest rates will be
lowered and the quantity of total media will be increased; if they are
rising, the opposite course will be pursued.

In order to study the behavior of the various factors in international
payments and to determine whether they will respond to, or will inter-
fere with, this mechanism, it is necessary to isolate each factor. Let us
first consider visible trade by itself. Assuming that this is the only
factor in such payments and that gold is not used, it is obviously neces-
sary that the in and out payments of each country on account of ex-
ports and imports should balance. For, in these conditions, there
would be no other means of payment. Under the strict notion of pur-
chasing-power parity, as it is often interpreted, if exchanges were at
this parity no good would move either way. Practically, however, the
scatter in the price complex is not the same in any two countries. The
effective equilibrium point in exchange rates—with reference to visible
trade only—is, therefore, not so much the point of purchasing-power
parity in terms of a global price concept as it is that point at which the
domestic values of aggregate imports and of aggregate exports are the
same. This may be called the equilibrium point of reciprocal purchas-
ing-power disparity in terms of the individual prices of international
goods.

Now if—under the terms of our analysis—internal prices were uni-
formly to rise or fall while foreign prices remained stable, the price
of foreign monies (i.e., exchange rates in terms of home money) would
rise or fall by exactly the same proportion. For when, for instance,
internal prices declined 10 per cent, the tendency, if exchange rates
remained unchanged, would not be to increase the quantity, and per-
haps the aggregate value, of exports but to decrease both the quantity
and the aggregate value of imports. But both of these changes would
tend to produce a greater supply of, than demand for, foreign money,
so that the price of foreign money would fall and thus restore the
balance. These two tendencies would exactly counteract each other.
The result is that the quantities of exports and imports would remain
the same if the scatter of the price complexes concerned remained the
same. Exports would be priced 10 per cent lower at home because of
the lower internal price level, but at the same prices abroad, because the
decline in price of the goods, would be exactly offset by the decline in
price of the foreign money; imports would be priced at the same level

abroad but lower at home because of the decline in the price of the foreign money. This automatic adjustment of exchange rates to uniform changes for a whole price complex would operate in exactly the same way when internal prices were stable and external prices were rising or falling. This generalization is but a modified statement of the doctrine of purchasing-power parity. While the latter is not acceptable as a complete definition of the equilibrium point in exchanges, where other factors than visible trade enter into international payments, it is certainly complete for the conditions which we have set at this stage of the analysis.

It is clear then that, if the only payments were on account of visible trade, there would be no external influences upon internal prices except those which occurred by reason of changes in the scatter within the price complexes at home and abroad. Therefore, if the mechanism for internal price control were effective, it might operate to induce or to prevent rises or falls in domestic price levels (and the corresponding rises and falls produced in exchange rates)without any interfering repercussions. And, since similar changes in foreign prices would exhaust their effect in producing corresponding fluctuations in exchange rates, it would be unnecessary for the mechanism to be brought into use to neutralize foreign price changes.

In a world in which only goods moved internationally and price changes were uniform within each complex, no international monetary standards or medium of payment would be needed, and the objective of internal price stabilization would not be interfered with by external causes nor by repercussions of monetary management. Some agency would be required to level out the day-to-day and the seasonal exchange irregularities, i.e., to offset fluctuations in exchange rates not due to changes in external or internal price levels as a whole or in the scatter within price complexes, but arising from the irregularities in daily in and out payments for an annually balanced trade. Beyond this, exchange rates could be allowed to find their own equilibrium points. Such freedom of exchange rates would have the additional advantage that it would counteract any artificial interference with prices or the movement of goods by means of tariff changes or other trade barriers. For the foreign price of the money of any country impeding its imports would automatically rise and thus either neutralize its artificial price changes or reduce its exports, or both.

However, when we introduce the other factors in international payments into this analysis, the results are very different. In so far as such in and out payments for all other purposes compensated each other (i.e., so long as the payments in each direction were equal to those in the other) and did not alter the relation between investment and sav-

ing at either end, they would not interfere with, nor disturb, the previous conclusions. But, in so far as there was an excess one way or the other in these payments, then, since we are assuming the absence of gold movements, this excess payment could only be made by altering the equilibrium point of reciprocal purchasing-power disparity (as to goods) until a sufficient alteration of, and discrepancy between, the in and out payments for visible trade was secured to permit and convey this excess. Let us suppose, for instance, the following situation to exist in the case of a country whose only financial relations with the rest of the world arise from visible trade.

Internal Price Level	Home Value of Exports	Home Value of Imports	
$1.00	1000 @ $1.00 = $1000.00	1000 @ $1.00 = $1000.00	

Exchange Rate	Foreign Value of these Exports	Foreign Value of these Imports	Foreign Price Level
$1 = £⅕	£200	£200	£⅕

If now we introduce other factors which give rise to an excess out payment of $110.00 which must be covered by a change in the previous balance of visible trade, then it is necessary that the picture be altered to something like the following:

Internal Price Level	Home Value of Exports	Home Value of Imports	
$1.00	1100 @ $1.00 = $1100.00	900 @ $1.10 = $990.00	

Exchange Rate	Foreign Value of these Exports	Foreign Value of these Imports	Foreign Price Level
$1.10 = £⅕	£200	£180	£⅕

The home difference between in and out payments on account of visible trade will now be $110.00, for which foreign funds of £20 will be obtained. This difference will convey the excess out payment on account of other factors.

Obviously, the introduction into the analysis of such excesses in the other factors in international payments requires the distortion of the exchange rates away from the previous equilibrium point of reciprocal purchasing-power disparity which formerly balanced the domestic values of exports and imports. But, if this excess payment is continuous and stable, a new equilibrium point will be reached at which the reciprocal purchasing power disparity induces just this necesary difference between the domestic values of exports and of imports.[1] Thereafter, the

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[1] The time patterns of the recent British and American depreciations should be valuable checks on this abstract analysis; however, because both were pathological cases and because both countries had just previously been on the gold basis, these examples are not in full accord with the conditions assumed in the

conclusions we have already arrived at with reference to the case of visible trade alone are equally applicable when combined with such established, stable, and continuous excess payments arising from other factors. After such an adjustment had once been accomplished and as long as this excess remained the same in amount and direction, the introduction of all the rest of the factors which enter into international payments—except gold—would still leave the objective of internal price stabilization insulated from the effects of uniform changes in external price levels and from indirect repercussions of monetary management at home.[2]

There remains for consideration, however, the question whether these conclusions are also true during the periods when changes in the volume or direction of any of the factors in international payments are inducing compensatory movements in the opposite direction through the medium of an alteration in the equilibrium point of the exchange

analysis. In both cases the immediate net "flight" of capital took place in the form of gold in the month or two preceding the suspension of gold payments. In both cases the early depreciation in the exchanges seems to have been caused, not by further net movements of capital, but by a radical increase in imports— a result of speculative buying—which occurred in the first 3 to 5 months "off gold," so that the adverse change in the balance of trade and the depreciation both reached their "natural" maxima about the third month. In both cases this depreciation apparently induced a large net influx of capital in the first 3 or 4 months which covered these increased imports plus any further movement of capital out and slowed down the depreciation. In both cases this influx seems to have consisted of induced repayments by foreign debtors desiring to take advantage of the depreciation, of speculative purchases of securities by foreigners in anticipation of inflation, and in repatriation of the previous and anticipatory flight which had taken place in the form of gold and in other ways (i.e., at least a partial liquidation of the short banking and speculative position in the exchanges). In both cases the effect on visible trade expected by theory—that of increasing exports and restricting imports—did not begin to be operative until the 6th month. Thereafter, in England, the capital export having ceased, the favorable shift in the balance of trade began to be covered by gold purchases, and in America it was covered at first by a secondary flight of capital and later also by gold purchases. Thus, in both cases, when the capital export had ceased, the natural tendency of the exchanges to rise again was nullified, and the depreciation was maintained or increased by somewhat artificial means. The only time that the capital export, which was the original depreciating influence, seems to have conveyed itself by means of an induced favorable balance of trade was during the secondary flight from the United States.

[2] This conclusion must be qualified by reason of the fact that even uniform changes in internal or external price levels, while tending, so far as visible trade alone was concerned, to produce a new equilibrium point in the exchange rates, without alteration in the balance of trade, nevertheless might thereby alter the amount of some of the other factors and thus create a change in the required distortion and so a change in the balance of trade. If they did so, then changes in price levels would not be so neutral as they seemed in the first instance.

rates. In other words, while other factors in international payments may not cause external interfering influences with internal price stabilization after the above described adjustments have been completed and a condition of equilibrium has been reached, it does not follow that this is also the case in the state of disequilibrium while the adjustments are in process. Since this is the crucial point in the analysis of international price relations in a world off gold, it is necessary, before proceeding, to establish certain distinctions as to the behavior of the various factors in international payments. The first distinction is between those factors which are, on the whole, regular and stable in volume and direction, and those which are not. The former tend to maintain existing equilibrium, the latter tend to disturb it. The second distinction is between those factors which are active in relation to the exchange rates and those which are passive—that is, those in which changes arise largely from causes other than alterations in exchange rates and those in which the changes are largely a response thereto. It is the active, or self-initiating, factors which tend to upset equilibrium; it is the passive, or induced, factors which tend to maintain or restore it. Finally, there is a class of self-compensating factors, changes in which are automatically cared for without effect on the exchange rates. In order to distinguish the principal disturbers of equilibrium from the rest, it is necessary to examine each of the major classes of international payments in the light of these three distinctions. Visible trade is, generally speaking, regular and passive when prices are stable. In fact, it is so largely the means by which compensation is secured for changes in the other factors, through the medium of altered exchange rates, that we have confined our previous example to this most usual method of compensation. However, changes within the price complex of international goods may be active and disturbing, particularly if they are the result of marked technological improvements or of marked improvements in the accessibility of natural resources. And uniform changes in relative price levels are active in their effect on exchange rates, though, as we have seen, they do not directly tend to disturb the balance of visible trade. They may, however, as suggested in the note above, by affecting the exchange rates, also affect the volume of other payments and thus require a new adjustment in the balance of trade. Invisible trade—the services and the remittances of tourists and immigrants—is a fairly regular and almost wholly active factor. So, too, is service on capital. Movements of banking, short-term, and speculative capital have, in the past, been highly irregular but also largely passive. They have constituted the most available and prompt method of compensation, being induced by comparatively slight changes in exchange rates and somewhat deliberately controlled

for this purpose by changes in relative interest rates. But this was in a world on gold. So far as we can judge by observation, these movements would become, in a world off gold, highly active as well as irregular. They would act as short-selling acts—enhancing the changes in exchange rates at the beginning of a fall or rise and only cushioning them as the turning points were approached. The class of what we have called self-compensating factors consists in part of movements of banking and short-term capital and in part of the next factor—movements of investment capital. When commercial credit (including banking) or investment capital (security issues) is used to finance specific exports (visible or even invisible), capital is exported and goods or services are exported in equal amount. No recourse is had to the exchange market. Therefore, though it be irregular, this type of capital export is not directly active. Changes in its amount or direction automatically compensate themselves. Nevertheless, this type of financing does set up, in the opposite direction, a potentially active effect on the exchanges which it does not compensate; for it increases, immediately or in the future, the incoming capital service factor. Again, this class of capital movement and many of the rest of the movements of investment capital introduce a special feature which must be considered. If the movement is in the form of credit (not ownerships), it may be stated in terms of the money of the lender, or of the borrower, or of a third country. In this respect it may enhance, at one or both ends, the active effect upon the exchanges of the capital service item which it gives rise to. Since it constitutes a need in the borrowing country for a fixed amount of the money of some other country, this influence will tend, once disequilibrium is produced, continually to increase it and to remove continually the point at which a new equilibrium can be established. The other types of movements of investment capital, which are not self-compensating, are not classifiable, as a whole, according to the first two distinctions. Some are fairly regular—England's to the Empire for instance; some are quite irregular—those of the United States to Germany, for instance; some are active—flights of capital, for instance; and some are comparatively passive—induced by the known possibility of increasing general exports, and thus taking the form of an advance compensation for a permitted change in the balance of trade without waiting for the effect of the thus avoided alteration in exchange rates.

We may now select from the factors in international payments, thus analyzed, the particular types which are disturbers of equilibrium because they are irregular, active, and not self-compensating. These are changes within the price complex of international goods; in some respects changes in relative price levels; changes in invisible trade and,

to some extent, in capital service; and those particular movements of capital of all kinds which are irregular, active, and, therefore, not self-compensating. For our purpose, perhaps the most typical sample for further examination will be a movement of capital induced by a difference in interest rates. The price effects of this one sample will be typical of all, except in one respect; we must make due allowance for the fact that a capital movement may produce changes in the investment-saving ratio and thus affect price stabilization in a way that the others will not do. Obviously, since the combination of several of these changes or movements, occurring at the same time, may fortuitously offset each other, we must, for the purpose of this analysis, assume that our sample capital movement is one which is active, that is, one which must convey itself by means of a change in the balance of visible trade.

The excess out payment corresponding to this capital movement, while the necessary induced adjustments in other factors were in process, would—in the absence of gold movements—have to affect first the exchange rates, thereby the demand for and supply of international goods both at home and abroad and finally, perhaps, the investment-saving ratio. In these two ways, both internal and external price levels might be influenced and stabilization at either end might be interfered with. Let us assume, at the start, for the sake of simplification, that internal and external price levels had at the time no other tendency to instability. Considering first the effects on the capital exporting country, a rise in the price of foreign monies (exchange rates) would raise the price of imports not displaced. In any country in which foreign raw materials were an important element in imports, this would tend to raise the whole price structure based on them. The additional demand for export goods would tend, so far as the supply was not completely elastic, to put up their prices. Since the export of capital involves no diminution in national income, demand for displaced import goods would continue and would be transferred to substitute home products. The prices of these might react in the same way. Finally, such an export of capital means the transfer, directly or indirectly, of idle funds from the hands of domestic investors to the hands of domestic producers without altering the rate of saving.[3] For, if this capital export had to be effected by an alteration in the balance of visible trade, then the exporting capitalists—in the last analysis, but indirectly—would have to buy their foreign money from the exporters of their own country, trading their own domestic money for the additional foreign monies then received by these exporters as well as for the foreign monies no longer required by the importers. In view of the

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[3] If the funds were in fact those of foreigners, who were repatriating them, the effect would be the same.

concurrent additional demand for output the effect of this conversion of idle into active funds—saving remaining the same—would be to increase I/S, and so produce another tendency to raise prices. If the funds which represented the exported capital were already active, since they were then only diverted from one active use into another, there would be no change in the I/S ratio.

In the capital importing country the effects, other things being equal, would be the reverse. In the absence of gold movements, there would be a rise in the exchange value of its money which would tend to lower the prices of its imports and, thus, of the price structure based on them. The pressure of the additional supply of imports, so far as it tended to displace domestic production, would tend to lower those prices as well. And the curtailment of exports would have the same effect on prices of exportable goods. If the capital imported under these conditions remained idle, since the funds obtained were those formerly going into domestic productive channels, the result would be a transfer of active into idle funds and a decrease in the investment-saving ratio, saving not being affected. This would also tend to lower price levels. If, however, the capital import went into new investment, the funds exchanged would merely be transferred from one to another active use and, therefore, would have no effect on this ratio.

So much for the effect of such a capital movement when "internal and external price levels have at the same time no other tendency to instability." In converting these conclusions to fit the realities of a world off gold, they need to be only slightly altered, as follows: The effect of an active and not accidentally or automatically compensated increase in out payments of any of the types described would be a *tendency* to rising prices at home and declining prices abroad in the field of international goods, and, so far as the investment-saving ratio was altered, in the field of general prices as well. If external prices were rising at the time, from other causes, they would rise less fast or the price of foreign monies would fall less fast because of the extra out payment— and *vice versa*. If internal prices were rising at the time from other causes, they and the prices of foreign monies would rise faster— and *vice versa*. But, if the assumption that the machinery for securing internal price stability will work effectively should turn out to be correct, this latter modification would be unnecessary.

The effect produced by these tendencies upon the supply of and demand for funds in each country must also be considered. To return to our chosen example, it is clear that, in the capital exporting country, while the diversion of idle capital to productive use might finance the production of the additional exports and even the production of the domestic goods substituted for former imports—depending on the rate

of turnover of circulating capital and the amount of new fixed capital needed—it would not be sufficient both for this and for a general rise of prices. There would therefore be a tendency, coming from the demand side, for credit and circulating media to increase in quantity. If the exported capital had not been idle, the shortage of credit would be greater still. On the other hand, in the importing country, while the importers might need no more funds, since they would take a larger quantity of goods but at a lower price in their money (in view of the exchange rate), the exporters would require less and would have surplus funds. Whether or not the funds indirectly furnished for the capital import, by the deprivation of the domestic producers of former goods now displaced by imports, were sufficient—and this would depend again on the producers' rate of turnover—it is clear that, when there was added the exporters' surplus funds and any funds released by a more or less general reduction of prices, the aggregate released would provide a sufficiency for the capital import. There would need to be no increase and there might even be a decrease in the demand for credit.

In these circumstances, could the management of money, attempting to maintain stable price levels at either end, be successful? In the capital exporting country the application of the formula of a rise in interest rates and a contraction in circulating media could only affect the process by one of two means. The rise in interest rates might attract sufficient foreign capital to offset the export of domestic capital. This was one of the standard expedients of the 19th century. It depended on freedom of movement, on confidence in the stability of exchange rates, and it established an unstable condition in international short-term indebtedness. Would it operate to correct the situation in a world off gold, or would it merely transfer the disequilibrium to some other country, not hitherto affected, by inducing a capital export therefrom? An induced compensatory movement confined to the capital-exporting and the capital-importing country is practically inconceivable. The only way in which the second means, the contraction of circulating media, could be effective would be by reducing national income sufficiently to release such part of the increment of export goods from home consumption, and to prevent the replacement of such part of the decrement in import goods which, in either case, could not be provided without a rise in prices, or to accomplish an equivalent curtailment of production elsewhere. So far as international goods were concerned, a rise in prices could thus be prevented. But even this rigorous program could not prevent a rise in the prices of goods which continued to be imported nor in the price structure based on them. The cost of even this degree of control of prices would then be, in effect,

to reduce domestic consumption by all or part of the amount of the capital export. Would that be politically feasible in the face of the prospect of "prosperity" which increased exports present? At best this technique would be slow in operation; it would not be likely to succeed in offsetting the illusion of a permanently increased demand; and it could not avoid the real reapportionments of purchasing power at home which had been engendered by a temporary situation.

In the capital-importing country a lowering of interest rates and an increase in circulating media might cause a compensating export of capital. Again, in a world off gold, would this be probable, and, if it did occur, would it not merely transfer the effect to some other country not previously affected? In the face of declining demand for credit and circulating media and a replacement of domestic goods by foreign goods in domestic consumption, the expansion of circulating media could only serve to maintain prices if it increased domestic consumption or increased investment in relation to saving by the necessary portion of the capital import. Probably it would only be operative in the latter way and that very tardily.

The results of this abstract analysis appear, at first sight, to be diametrically opposed both to customary theories and to actual observation. Nevertheless, within the strict pattern of the premises, the conclusions seem to be correct. And these premises seem to conform to the conditions which would exist in a world off gold. The fact is that the generally accepted theories of international price relations have been constructed upon the tacit assumption of a world using some general monetary means of international settlement, and, for the most part, that is the world we have observed. Like the old physicists, we have had a fixed point in the backs of our minds—the mechanism for fixing exchange rates—around which other things were conceived to oscillate, and it is difficult for us to reconstruct our mental images upon the basis of a world of complete relativity. Yet it is just such a world which many of the advocates of internal price stabilization are proposing. Perhaps they do not fully appreciate the fact and suppose they are only substituting one fixed point—a mechanism for stabilizing prices—for the other. But I think our analysis demonstrates that this is not the case.

It is possible to gain certain support for the abstract analysis from recent observation, and to some extent from existing theory. It seems to be generally agreed that the movements of capital out of Germany and France during the great inflations were partly effected—almost wholly so far as they were net movements—by inducing a more favorable balance of trade as the means for their own escape. The movement of capital continually depreciated the exchanges more rapidly than the

loss of internal purchasing power at that stage warranted, and thereby increased exports without increasing imports, or faster than these were increased. In turn, this excessive depreciation necessarily raised the internal prices of imported goods; the resulting increased foreign value and demand appear to have acted on the price of exportable goods; thus the prices of both classes of international goods were driven up faster than general inflation was raising the internal price of non-international goods.[4] We were not able to watch the effect on prices of an increased I/S ratio in these cases only because conditions were at the time so pathological that internal investment, if occurring at all, was probably not induced by the change in the balance of trade. But I think it would be admitted that the process of capital exports enabling an increase in visible exports and thereby affecting the I/S ratio was observable in the United States from 1924 to 1929. The fact that we were then on gold would not make this an inappropriate example because the internal chain of cause and effect is the same in both conditions. To be sure these are but partial proofs. Moreover they relate only to the effects on the capital-exporting country. Nevertheless, some weight must be given to reasoning *per contra* in support of the obverse of this process.

From the standpoint of theory there is also some confirmation. It is generally recognized that, when fixed exchange rates are maintained, prices tend to fall in the country in which internal purchasing power is relatively low; and *vice versa*. From this arises the corollary—a theory currently proclaimed and practiced—that, by altering the exchange rates, this otherwise effective influence may be eliminated. But such a deliberate lowering of exchange rates can only be procured, when payments are in balance and gold is not used, by means of inducing an export of more or less frightened capital. In effect this is a method, then,

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[4] See, for instance, A. Aftalion, *Monnaie, Prix et Change.* It may be estimated that some 20–25 milliards of capital in terms of francs Poincaré were exported from France during 1924, 1925, and the first 7 months of 1926, through the avenue of the induced favorable change in the balance of visible trade. This out of a total flight of 37 milliards for the period, according to Meynial's estimates. On the same basis, some 2 milliards of capital in terms of gold marks were exported from Germany during 1923 by the same means.

In the case of France, the comparatively large scale of the later return of domestic capital is evidence that the original flight of domestic capital had been of greater proportions than was that from Germany. From this we may conclude that, of the two influences producing the rise of prices—the flight of capital and inflation of the currency—,the former had more influence during the final period in France than it did in Germany; and that in France the currency lost external purchasing power more rapidly in comparison to its loss of internal purchasing power than it did in Germany. The indices seem to confirm this conclusion.

of sustaining a previously unnatural trade balance by exporting fright-
ened capital to carry it, and of correcting a previous disequilibrium in
reciprocal purchasing-power disparity by bringing about conditions
which cause the same trade balance to represent equilibrium. However,
this is an expedient which only works as long as the flight of capital
continues. While both this theory, and the practice based on it, go to
confirm that part of our conclusion which deals with the effect of capi-
tal movements in sustaining an otherwise falling price level in the capi-
tal-exporting country, or even raising it, the practice might be discon-
tinued if it could be made equally obvious that its effect only continues
during the continuance of the flight of capital and is even then being
continually diminished by the rest of our conclusion—the necessarily
depressing effect on prices in the country to which the capital is fleeing
—,an effect which must in the end defeat the original purpose.

The principal theoretical objections to our conclusions derive from
observations of conditions that are inapplicable. In the first place, it
should be noted that, in the theory at least of the 19th century gold
standard, movements of capital, which were not fortuitously or auto-
matically compensated, induced movements of gold. Their effect was
then to produce *automatically* exactly the contraction of credit in the
exporting country and expansion in the importing country that would
seem to be called for to correct the price effects which we have de-
scribed. But, when the movement did take place in gold, it did not take
place by means of a readjustment of the trade balance. Therefore, the
conditions requiring this contraction and expansion in order to main-
tain stable price levels were not produced. Since the movement of capi-
tal in the form of gold produced no tendency to a price rise in the ex-
porting country, nor to a fall in the importing, the gold movement,
through its effect on credit, necessarily tended to produce a decline—
not to offset a rise—in the exporting country and to produce a rise—
not to offset a decline—in the importing country. On the other hand,
when shifts in the balance of payments induced compensation in other
ways than by gold movements, what usually took place was a com-
pensating shift in the balance of visible trade. Since, under the gold
standard, exchange rates were fixed, the only way to accomplish this
was through an induced change in relative price levels. Because the
money of the exporting country was not allowed to fall, its prices were
required to fall instead. Or, because the money of the importing coun-
try was not allowed to rise, its prices had to rise instead. In one or the
other way, or in both, the relation of the two price levels was compelled
to alter itself. Thus, with fixed exchanges, the price effects of an active
shift in the balance of payments, whether the shift was compensated
by gold or by other induced means, were exactly the opposite of what

they would be with free exchanges in the absence of gold movements.

One other question suggests itself. Is the concept of capital movements, in general, a valid one for a world off gold? In respect of movements engendered by the possibility of speculative profits in exchanges or by the assumption of greater safety in conserving purchasing power, the experience of the post-war period shows that it is. And if, while still off gold, the regular in and out payments should come gradually and automatically into balance, then it seems probable that the resulting steadiness of exchange rates around the new equilibrium points of reciprocal purchasing-power disparity would soon tempt capital to move as well in response to differences in interest rates and of opportunity for profitable investment—at least until it was discouraged by resulting aberrations of exchange rates. Capital movements of the kind we have described would then seem to constitute payments which must be considered among the potentially upsetting factors even in a world off gold. Certainly the other active variations in direction or volume of payments, which we have enumerated, would continue under any conditions other than complete isolation. Their absolute prevention is obviously impracticable under modern conditions. Therefore, it seems necessary to take them into account as an unavoidable condition of the operation of any monetary system.

Our conclusions are disturbing. It appears that the choice between the two aims—stabilization of prices and stabilization of exchanges—is not as clear-cut as seems to have been supposed. If the only step taken was the sacrifice of stable exchanges, it appears that any single country seeking to attain the objective of stable internal prices by itself would need to isolate itself completely, both commercially and financially, from the rest of the world. For such *laissez-faire* with regard to exchange rates, alone, would not prevent interfering and almost uncontrollable repercussions arising from international economic relations. In the absence of a medium of international settlement, visible trade alone might be conducted without interference. But the requisite condition would be that all changes in the scatter within the price complexes due to technological or other causes were excluded. An unvarying volume of business in the various other factors in international payments might be added without interference. But the additional requisite condition would be that this volume, in turn, would not be affected by changes in the exchange rates induced by uniform changes in price levels at home or abroad. Finally, even a varying volume of business might be conducted without interference. But the additional requisite condition would then be that changes in the debits and credits were always made, fortuitously or through control, exactly to compensate each other, so that no change of balance needed to be induced among

the passive factors by means of alterations in the exchange rates. Of course, all these necessary conditions are impossible of attainment. The hope of autonomy in the stabilization of internal prices by the mere sacrifice of exchange stability is a vain delusion.

In the real world, the choice does not lie between two incompatible objectives—stable prices and stable exchanges—one of which must necessarily be sacrificed to reach the other, and the sacrifice of one of which will enable us to reach the other. On the contrary, complete freedom to pursue the objective of stabilization of internal prices requires, as its major premise, that all fluctuations of exchange rates produced by any other influence than a uniform rise or fall of prices abroad shall be prevented. Then—but then only—would a change in parities conduce to the maintenance of stability in internal prices. In practice this ideal could never be reached. For every permanent change in the volume or direction of payments, which was not automatically or accidentally compensated, would either require an adjustment of exchange parities, and thus influence internal prices indirectly, or it would influence them directly by compelling a change in the relation between external and internal prices to secure the necessary distortion. Therefore, in practice, the actual choice lies between various methods of stabilizing exchange rates, or, in other words, between different monetary mechanisms for international settlements. To judge from our analysis, in the absence of a generally accepted medium of payment, international commerce and finance, as it actually behaves, would result in a constantly changing equilibrium point in reciprocal purchasing-power disparities which, in turn, would constantly bring external influences to bear upon internal prices in a way that no management could offset. On the other hand, if such commerce and finance were conducted with an international medium of payment, this could be used within its capacity to compensate at least the temporary irregularities between aggregate in and out payments and thus, at most times, prevent them from inducing other compensations in a way which would affect prices. It is the use—not the disuse—of an international money which seems to be required for even the partial success of a program of internal stabilization. The first essential seems to be that the country should be on gold in some form (or its equivalent) not off gold.

In the past, to be on gold signified a readiness to buy or sell gold at a fixed price, at least for use in international settlements and at least within the limits which, it was conceived, could safely be added to or subtracted from the gold stock. When these limits were exceeded, a country went off gold—that is, ceased to buy or sell it—and the exchanges were turned loose to find their natural equilibrium points un-

der, of course, the most abnormal and, therefore, unbalanced conditions. Under such conditions our analysis indicates how fruitless would be the efforts to stabilize internal prices. Yet it has been precisely at such times that the program of internal stabilization has been urged most strongly in each country. However, the use of gold for international settlements does not necessarily imply a fixed buying and selling price. In the last two years England has conducted an interesting experiment along new lines, purchasing gold as a commodity in an open market and at varying prices for the special purpose of covering mercurial movements of its own and of foreign capital. The result has been compartively stable rates for British exchange, which indicates the comparative success of the expedient. The weight of the pressure of payments requiring compensation has been removed from the exchanges, and, therefore, from the balance of visible trade and, therefore, from the internal price level, by the interposition of gold movements. It may prove that the great monetary inventor, England, has, in this respect, hit upon a new method of using gold in balancing international payments which will become part of an improved gold standard for the future.

Some international monetary system seems required to permit even a reasonable degree of deliberate price stabilization anywhere. In the past, we had an international monetary system. It functioned within limits in settling international balances and, often, in unsettling international prices. It established a rigidity in exchange rates which permitted the artificial falsification of the existing parity by means of trade barriers applied *ad absurdum*, and did not permit the rectification of the parity either to offset these artificial distortions or to correct for natural and permanent changes. When it ceased to function there was chaos. International money no longer moved. Then there was no monetary compensation whatever and exchanges ran wild. Recognition of the necessity of an international monetary medium and, at the same time, recognition of the incompatibility of the old system with present conditions and with new ideals, suggests that, perhaps, in the future, by experiment and experience, we may learn the technique of a more elastic method of pricing gold which will gain for us certain notable advantages—a greater degree of freedom, for those of us who desire it, in securing stability of internal price levels than was possible under a fixed price for gold; a greater liberty of action in nullifying the arbitrary leverage of trade barriers than was possible with fixed exchange rates; and a system of settlements which will continue—at a price—to temper abnormal fluctuations in exchange rates even after disequilibrium has reached the old point of a break-down and only the gradual subsidence of fluctuations at a new and natural equilibrium point for

the exchanges will succeed in securing the necessary reciprocal pur-
chasing-power disparity to bring in and out payments into balance
without further movements of gold. The essential of such a method of
pricing gold is that it should not make gold movements an active fac-
tor nor itself act as a disturber of equilibrium.

THE MAXIMUM VALUE OF URBAN LAND CONVERTED TO DIVERSE USES

By Henry A. Babcock

I. INTRODUCTION AND CONCLUSIONS

IN THIS paper, an attempt is made to find a starting point for a comprehensive theory of urban land utilization and city growth in its broader aspects. The paper treats of the subject in mathematical terms, starting with certain assumptions as to the demands for land at various price levels in a growing city and proceeding to show the general characteristics of city development which result from the play of economic forces. The paper describes, in general terms, the city pattern and the spread of building development which result from the fact that each owner of land will, in most cases, seek to obtain the greatest possible value by the sale or utilization of his land.

An increase of population brings about demands for additional land to serve as sites for additional commercial, industrial, residential, and social use buildings. Certain locations and sites are more desirable than others and, further, the intended use is a factor in determining the purchase price. For example, the prospective purchaser of a site for a legitimate theatre in the central business district of a city cannot pay so high a unit price for land as the prospective purchaser of a site for an office building.

Starting with a tract of land, or an aggregate of parcels of land, all in the same ownership, and assuming that there is a constant annual demand for this land for many diverse uses and, therefore, at different price levels, it is shown that the owner will realize the maximum value by accepting offers at all the prices from the highest down to a certain price, called the *critical price*, and by rejecting all offers at prices lower than this critical price. This means that he will realize a greater value by selling some land each year at each of the various price levels above a certain level than he will by selling it all at any one price, high or low, regardless of how slowly or rapidly he may be able to dispose of it. The critical price depends on the demand rates, the range of offered prices, and the original area involved. Methods are described in this paper for determining the critical price under various sets of conditions.

It is also shown that, for a maximum value, the owner will reject, after a certain time, further demands at the critical price, and after a

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longer time, will reject demands at the next higher price level, and so on, until finally he sells the last piece at the highest price. At the same time that he is rejecting demands at the lower prices, he is holding land vacant to be sold later at a higher price. Methods are developed for calculating the allocation of the total original area among the various price groups.

Now, these rejected demands are filled somewhere outside of the original area considered, and if it may be assumed that prospective purchasers in all price groups, regardless of how low, prefer land in the original area to land outside that area and accept the latter only because the owner of the original area will not sell to them at their prices, or, what is the same thing, only because they cannot secure land in the original area at a sufficiently low price, and assuming, for the moment, that the "outside" land is in one ownership, then, it is shown that, by accepting in a certain definite and determinable manner the offers rejected by the first owner, the second owner can produce a maximum value of his area. Methods are described for calculating the allocation of this second area among the various price groups. The second owner, under the stated conditions, will accept immediately all offers at prices lower than the first owner's critical price (but not lower than his own critical price), because these are the only opportunities he has to sell until the first owner begins to reject offers at successively higher prices, as previously described. Ultimately, of course, the first owner is sold out, and then the second owner will begin to benefit from the demand at the highest prices. As before, the second owner will reject, from the beginning, demands at prices lower than his critical price, which, in turn, become available to the owner of a third area, and so on.

It is further shown that the allocation of land among the various price groups which results when the first and second owner are in competition, is precisely the same as it would be if both tracts had been in the same ownership from the beginning and the same thing is true for the first, second and third areas, and so on. This is not only a startling result, it is one of great significance. It removes the original limitations on the theory brought about by assuming that all of the most desirable land had to be in the same ownership to make the results valid. It also shows that a city would develop in the same way if all the land were in a single ownership as it would in multiple ownership, except for one thing, and that is, that in multiple ownership there may not be agreement between owners' estimates of future demands for land and it is the expected future demand which controls the degree of spreading of building development.

Another consequence, under the stated conditions, is that parcels of land sold at different unit prices at the same time, exist side by side,

and that, while the "highest uses" tend to concentrate in the most desirable locations, it takes a long time to force out all of the lower uses. Witness the central business district of any American city. A corollary of this proposition is that while, very broadly speaking, land values are lower and lower at greater and greater distances from the most desirable locations, there is no uniformity of land values in a district and cannot be, for the economic reasons stated. This is of considerable significance in setting up an adequate and proper system of levying ad valorem property taxes.

Still another consequence is that it is not economically feasible to have all the highest class office buildings adjacent to one another, nor any other segregation of that kind. The theory accounts for the heterogeneous building pattern of cities. Also, it shows that it is not economically feasible for a city to grow solidly outward from the center, layer after layer, as an onion or a pearl does.

In Los Angeles, where, until recently, there was a widely held belief that the population would continue to increase at a very rapid rate, the building development has been pushed out to a very great extent and a very large percentage of vacant land left behind. On the other hand, in Baltimore, a much less rapidly growing city, the building development is much more compact and the left-behind vacant land is a much smaller percentage of the total.

In anticipation of criticism of the fundamental assumption of this analysis, that is, that the annual demands for land at the various price levels are each continuous and constant, it may be said that all that has been attempted so far is to gain a starting point for a much more comprehensive theory which will take account of variations in these demand rates.

Two paths are available for further development. On the theoretical side, analyses may be made under the assumptions of (1) declining rates of demand, (2) cyclical rates of demand, and (3) intermittent rates of demand. On the practical side, statistics can be amassed dealing with unit prices, the demands for land at these prices, and the types, amounts, and distribution, of land utilization.

The practical importance of such data, coupled with a comprehensive theory, to those concerned with the sociological, economic, and municipal problems of the growing city, is very great.

II. ALLOCATION OF LAND TO DIVERSE USES TO PRODUCE MAXIMUM VALUE

Assume a constant area of land, A, for which there is a continuous demand on the part of prospective purchasers, some of whom wish to use parts of it for some particular use, and others of whom wish to

use parts of it for other uses. Assume that these various prospective purchasers, each having a particular utilization in mind, are willing and able to pay per unit area the amounts $w_1, w_2, \cdots, w_s$, respectively, depending on the particular use to which each such purchaser intends to put that portion of the land which he is willing to buy. Assume, further, that the demand for this land by those purchasers who are willing to pay w_1 per unit area continues at the area-time rate, R_1 (sq. ft. per year), that the demand for this land by those purchasers who are willing to pay w_2 per unit area, continues at the area-time rate, R_2, etc. Assume, for the moment, that the entire area , A, is in single ownership.

This owner, therefore, under the assumptions stated, has an opportunity to sell his land, parcel by parcel, over a period of time. If he sells it all at the unit price w_1, his income from sales per unit time will be $(R_1 w_1)$ and will continue as an annuity for $A/R_1 = N$ years. The initial value of this prospective sales income, subject to an ad valorem tax on the unsold portion at the rate ρ on full value, will be[1]

$$(1) \qquad\qquad V_0 = R_1 w_1 a_{\overline{N}|i+\rho},$$

where $a_{\overline{N}|\,i+\rho} = $ present value of 1 per annum for N years at the interest rate $(i+\rho)$; and where $i = $ rate of return on value.

If the owner sells at the rate R_1 for N_1 years at the price w_1 and at at the same time sells at the rate R_2 for N_2 years at the price w_2, and so on, the initial value of his prospective income, subject to an ad valorem tax on the unsold portion, as before, will be

$$(2) \qquad V_0 = R_1 w_1 a_{\overline{N_1}|} + R_2 w_2 a_{\overline{N_2}|} + \cdots + R_s w_s a_{\overline{N_s}|},$$

subject to the condition that

$$(3) \qquad\qquad R_1 N_1 + R_2 N_2 + \cdots + R_s N_s = A.$$

The value, V_0, will obviously be different as different values of the N's are chosen, but will be a maximum with one particular set of N's.

Since $a_{\overline{N}|i+\rho} = \dfrac{1 - u^N}{i + \rho}$, where $u \equiv \dfrac{1}{1 + i + \rho}$, (2) may be written

$$(4) \qquad \begin{aligned} V_0 = {} & \frac{R_1 w_1}{i + \rho} (1 - u^{N_1}) + \frac{R_2 w_2}{i + \rho} (1 - u^{N_2}) \\ & + \cdots + \frac{R_s w_s}{i + \rho} (1 - u^{N_s}). \end{aligned}$$

[1] *American Mathematical Monthly,* XL, No. 3. March, 1933, 151, Eq. (18.1); "On the Valuation of Land Awaiting Conversion to a Higher Use," by H. A. Babcock.

The total differential of V_0 is

$$(5) \quad dV_0 = -\frac{\log u}{i + \rho} (R_1 w_1 u^{N_1} dN_1 + R_2 w_2 u^{N_2} dN_2$$
$$+ \cdots + R_s w_s u^{N_s} dN_s).$$

The total differential of A, in (3), is zero, because A is a constant, hence,

$$(6) \quad dA = R_1 dN_1 + R_2 dN_2 + \cdots + R_s dN_s = 0,$$

and

$$(7) \quad dN_1 = -\frac{1}{R_1} (R_2 dN_2 + R_3 dN_3 + \cdots + R_s dN_s).$$

This value of dN_1, substituted in (5), gives

$$(8) \quad dV_0 = \frac{\log u}{i + \rho} \left[R_2(w_1 u^{N_1} - w_2 u^{N_2}) dN_2 + R_3(w_1 u^{N_1} - w_3 u^{N_3}) dN_3 \right.$$
$$\left. + \cdots + R_s(w_1 u^{N_1} - w_s u^{N_s}) dN_s \right].$$

For the maximum value of V_0, it is necessary that the coefficients of $dN_2, dN_3, \cdots, dN_s$, each be zero. Since, by tacit hypothesis, the R's are different from zero, and $\log u / i + \rho$ is different from zero for all finite values of $(i + \rho)$, including zero, it follows that, for a maximum V_0,

$$(9) \quad w_1 u^{N_1} = w_2 u^{N_2} = w_3 u^{N_3} = \cdots = w_s u^{N_s},$$

and, therefore,

(9a) $\log w_1 + N_1 \log u = \log w_2 + N_2 \log u = \cdots = \log w_s + N_s \log u.$

From (9a)

$$(10) \quad N_1 - N_\eta = \frac{\log \dfrac{w_\eta}{w_1}}{\log u} \equiv c_\eta, \quad (\eta = 1, 2, \cdots, s).$$

Note that

$$c_1 = N_1 - N_1 = \frac{\log \dfrac{w_1}{w_1}}{\log u} = 0.$$

The c's, as defined by (10), are expressed in terms of the data: $w_1, \cdots, w_s$, and $(i + \rho)$, and are independent of the R's. There are $(s-1)$ simultaneous equations in (10) between the s unknown N's, but (3) supplies the remaining relationship necessary for the solution. The s equations involving the N's may be written

$$(11) \quad \begin{cases} R_1N_1 + R_2N_2 + \cdots + R_{\eta}N_{\eta} + \cdots + R_sN_s = A. \\ N_1 - N_2 \qquad\qquad\qquad\qquad\qquad = c_2. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ N_1 \qquad\qquad - N_{\eta} \qquad\qquad\qquad = c_{\eta}. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ N_1 \qquad\qquad\qquad\qquad - N_s = c_s. \end{cases}$$

The solution may be accomplished by multiplying the first equation by unity, the second by R_2, the third by R_3, and so on, and adding the resulting equations. Hence,

$$N_1 \sum_{\eta=1}^{\eta=s} R_{\eta} = A + \sum_{\eta=2}^{\eta=s} c_{\eta}R_{\eta}.$$

Since $c_1 = 0$, by (10), this equation may be written

$$(12) \quad N_1 = \frac{A + \sum_1^s c_{\eta}R_{\eta}}{\sum_1^s R_{\eta}},$$

and, by (10),

$$(13) \quad N_{\eta} = N_1 - c_{\eta} = \frac{A + \sum_1^s c_{\eta}R_{\eta}}{\sum_1^s R_{\eta}} - c_{\eta}.$$

By use of (13), the original area, A, can be so divided and the parts allocated to be sold at the various available prices, $w_1, \cdots, w_s$, in response to the annual demands, $R_1, \cdots, R_s$, as to produce the *maximum* value of the area. Since, for maximum value, the owner will accept offers at the price w_{η} for only N_{η} years in response to the annual demand R_{η}, it follows that he will sell an area $N_{\eta}R_{\eta}$ at this price. This area may be designated A_{η}. Therefore,

$$(14) \quad A_{\eta} = N_{\eta}R_{\eta} = R_{\eta}(N_1 - c_{\eta}) = R_{\eta}\left[\frac{A + \sum_1^s c_{\eta}R_{\eta}}{\sum_1^s R_{\eta}}\right] - c_{\eta}.$$

By (3)

$$(15) \quad \sum_1^s A_{\eta} = A.$$

An examination of (13) and (14) will show that, if $c_\eta > N_1$, A_η and N_η will be negative. From the nature of the problem, "negative areas" and the acceptance of offers for a "negative time" are inadmissible. It becomes necessary, therefore, to examine further into this question.

Without any loss of generality, the unit prices may be so numbered by subscripts that

$$(16) \qquad w_1 > w_2 > w_3 > \cdots > w_s.$$

It should be noted that the smaller the value of a "w," the larger is the value of the corresponding "c," by (10). Since u is less than unity and hence has a negative logarithm, and since u is constant, it follows that

$$(17) \qquad c_2 < c_3 < \cdots < c_\eta < \cdots < c_s,$$

and that all the c's are positive. For any given set of data, A, $(i + \rho)$, $w_1, \cdots, w_s, R_1, \cdots, R_s$; N_1, as given by (12), is a constant. Therefore, by (13) and (17),

$$(18) \qquad N_1 > N_2 > \cdots > N_\eta > \cdots > N_s.$$

The original value equation, (2), states that the total initial value is made up of a series of terms, $R_\eta w_\eta A_{\overline{N\eta}|}$, but it is not possible, because of the physical nature of the problem, to include any terms in which N_η is negative nor is it necessary to include any term in which N_η is zero. In the set $N_1, \cdots, N_s$, let N_{j+1} be zero or the first negative N in case there is no zero value. Then, from (18), it follows that all N's with subscripts ranging from 1 to j inclusive are positive and that the remaining N's with subscripts ranging from $(j+1)$ to s inclusive are negative (except that N_{j+1} may be zero). Therefore, as a practical matter, the upper limit in the summations appearing in (13) and (15) cannot be greater than j, and all price groups lower than w_j must be excluded. This price, w_j, may be called the "critical price."

The value of N_1 (and hence N_η) is a function of the upper limit in the summations; therefore, the value of N_1 computed by taking j as the upper limit will in general be different from the value of N_1 computed by taking s as the upper limit. If, for example, h is the upper limit, N_1 is a function of h, which may be written

$$(19) \qquad N_1(h) = \frac{A + \sum_1^h c_\eta R_\eta}{\sum_1^h R_\eta}. \qquad \text{Cf. (12)}$$

Similarly, if $(h+1)$ is the upper limit,

$$(20) \qquad N_1(h+1) = \frac{A + \sum_1^{h+1} c_\eta R_\eta}{\sum_1^{h+1} R_\eta}.$$

To determine the conditions under which $N_1(h) = N_1(h+1)$, equate the right hand members of (19) and (20).

$$(21) \qquad \frac{A + \sum_1^h c_\eta R_\eta}{\sum_1^h R_\eta} = \frac{A + \sum_1^h c_\eta R_\eta + c_{h+1} R_{h+1}}{\sum_1^h R_\eta + R_{h+1}}.$$

From (21), it follows that

$$A + \sum_1^h c_\eta R_\eta = c_{h+1} \sum_1^h R_\eta \qquad\qquad \text{and}$$

$$(22) \qquad A = \sum_1^h (c_{h+1} - c_\eta) R_\eta.$$

From (3),

$$A = \sum_1^{h+1} N_\eta(h+1) R_\eta = \sum_1^h N_\eta(h+1) R_\eta + N_{h+1}(h+1) R_{h+1},$$

and from (10),

$$(c_{h+1} - c_\eta) = N_\eta(h+1) - N_{h+1}(h+1).$$

By substitution of these values in (22),

$$\sum_1^h N_\eta(h+1) R_\eta = N_{h+1}(h+1) R_{h+1}$$

$$= \sum_1^h N_\eta(h+1) R_\eta - N_{h+1}(h+1) \sum_1^h R_\eta,$$

or,

$$(23) \qquad N_{h+1}(h+1) R_{h+1} = - N_{h+1}(h+1) \sum_1^h R_\eta.$$

For (23) to be true, it is necessary that $N_{h+1}(h+1)$ be zero, because the R's are all positive. Therefore,

(24)
$$
\begin{cases}
\text{If } N_{h+1}(h+1) = 0, \ N_1(h) = N_1(h+1). \\[2mm]
\text{If } N_{h+1}(h+1) > 0, \ N_1(h) > N_1(h+1), \ \text{because} \ R_{h+1} > - \sum_1^h R_\eta. \\[2mm]
\text{If } N_{h+1}(h+1) < 0, \ N_1(h) < N_1(h+1), \ \text{because} \ - R_{h+1} < \sum_1^h 'R_\eta.
\end{cases}
$$

Since, by the hypothesis that w_j is the critical price, N_{j+1} is either zero or negative and N_j is positive, it follows from (24) that

(25)
$$
\begin{aligned}
N_1(1) > N_1(2) > \cdots > N_1(j-1) > N_1(j) \\
\leqq N_1(j+1) < \cdots < N_1(s).
\end{aligned}
$$

From (25) it follows that $N_1(j)$ is the smallest of the N_1's, or, if there are two smallest N_1's, then these two are $N_1(j)$ and $N_1(j+1)$ respectively. This relationship makes it possible to determine the critical price (for any area A subject to the demands $R_1, \cdots, R_s$, respectively, the rate of return being i and the tax rate ρ) by computing successive values of N_1 by successively increasing the upper limit in (12) until the smallest (or first smallest) value of N_1 is determined. The upper limit of the summations which gives the smallest (or first smallest) N_1 is the subscript of the critical price.

Another important relationship involving the subscript of the critical price may be derived from (10) as follows

$$
N_1(j) - N_j(j) = c_j \qquad \text{and} \qquad N_1(j) - N_{j+1}(j) = c_{j+1}.
$$

Now, if w_j is the critical price, $N_j(j) > 0$ and $N_{j+1}(j) \leqq 0$. Therefore, $N_1(j) > c_j$ and $N_1(j) \leqq c_{j+1}$. Therefore,

(26)
$$
c_{j+1} \geqq N_1(j) > c_j.
$$

Under certain conditions the critical price will not fall within the range of prices $w_1, \cdots, w_s$, that is, the critical price may be lower than w_s. If, however, the critical price, w_j, falls within the range $w_1, \cdots, w_s$, the question arises as to whether or not the value of the area computed with price groups, $1, \cdots, j$, is greater than with some other group of available prices. Eq. (2), the value equation, may be written, with j as the upper limit in the summation (using the notation $V_0(j)$ to indicate that the value is a function of the upper limit), as

(27)
$$
V_0(j) = \sum_1^j R_\eta w_\eta a_{\overline{N_\eta}|} = \frac{1}{i+\rho} \sum_1^j R_\eta w_\eta (1 - u^{N_\eta}),
$$

in which all values of N_η are positive and w_j is the critical price, and in which the values of N_η are given by the relationship

$$(28) \qquad N_\eta(j) = \frac{A + \sum_1^j c_\eta R_\eta}{\sum_1^j R_\eta} - c_\eta = N_1(j) - c_\eta. \qquad \text{Cf. (13).}$$

Equations (27) and (28) give the mathematical and practical maximum value of V_0 so far as the price and demand set, $1, \cdots, j$, is concerned, but it has not yet been shown that $V_0(j)$ is greater than the value of any other selection of price and demand groups which can be made without introducing negative values of N_η. For example, suppose that that demand R_h at the price w_h, where h is any subscript from 1 to j inclusive, is arbitrarily rejected from the set $R_1, \cdots, R_s, w_1, \cdots, w_s$, and that thereby the critical price is changed from w_j to w_k. The question then is whether the value of $V_0(j)$ is greater than the maximum value of this new set: $R_1, \cdots, R_{h-1}, R_{h+1}, \cdots, R_s, w_1, \cdots, w_{h-1}, w_{h+1}, \cdots, w_s$. Using primes (') to designate the quantities in this new set and to indicate that one of the price groups has been dropped arbitrarily, the value equation is

$$(29) \qquad V_0'(k) = \sum_1^k R_\eta w_\eta a_{\overline{N_\eta'}|} - R_h w_h a_{\overline{N_h}|}, \qquad \text{Cf. (2),}$$

subject to the condition that

$$(30) \qquad \sum_1^k R_\eta N_\eta' - R_h N_h' = A. \qquad \text{Cf. (3).}$$

Obviously, $k \geqq h$. For a maximum value of $V_0'(k)$, following the procedure previously given, it may be shown that

$$(31) \qquad N_1'(k) = \frac{A + \sum_1^k c_\eta R_\eta - c_h R_h}{\sum_1^k R_\eta - R_h}.$$

$$(32) \qquad N_\eta'(k) = N_1'(k) - c_\eta.$$

$h \neq 1.$

$$(33) \qquad N_2'(k) = \frac{A + \sum_2^k c_\eta R_\eta}{\sum_2^k R_\eta} - c_2.$$

$$(34) \qquad N_\eta'(k) = N_2'(k) - (c_\eta - c_2).$$

$h = 1.$

It can be shown that k cannot be less than j, as follows: Placing $s = k$ in (12),

$$N_1(k) = \frac{A + \sum_1^k c_\eta R_\eta}{\sum_1^k R_\eta},$$

and the value of $(A + \sum_1^k c_\eta R_\eta)$ from the above equation substituted in (31), $(h \neq 1)$, gives

$$N_1'(k) = \frac{N_1(k) - c_h R_h}{\sum_1^k R_\eta - R_h},$$

which may be written

(35) $$\left[N_1'(k) - N_1(k)\right] \sum_1^k R_\eta = \left[N_1'(k) - c_h\right]R_h.$$

Now, the R's are all positive, and $N_1'(k) > c_k$ by (26), because w_k is the critical price of the set: $R_1, \cdots, R_{h-1}, R_{h+1}, \cdots, R_s, w_1, \cdots, w_{h-1}, w_{h+1}, \cdots, w_s, h \leq j$. Furthermore, $c_k \geq c_h$ by (17), therefore $N_1'(k) > c_k \geq c_h$. It follows that $\left[N_1'(k) - c_h\right]$ in (35) is positive, and therefore $\left[N_1'(k) - N_1(k)\right]$ is positive, or

(36) $$N_1'(k) > N_1(k).$$

By (25), $N(j)$ is the smallest (or one of two smallest) values of $N_1(\eta)$, therefore,

(37) $$N_1(k) \geq N_1(j),$$

whatever the value of k may be. For the second set, (26) becomes

(38) $$c_{k+1} \geq N_1'(k) > c_k.$$

Combining (36), (37), (38), and (26),

$$c_{k+1} \geq N_1'(k) > N_1(k) \geq N_1(j) > c_j.$$

Therefore,

(39) $$c_{k+1} > c_j.$$

(39) is true for any value of k not less than j, by (17), but is *not* true for any values of k less than j. When $h = 1$ the same procedure can be followed, using (33) instead of (31) and using $\left[N_2'(k) + c_2\right]$ instead of $N_1'(k)$, and noting that $c_1 = 0$. The result is the same and, therefore,

(40) $$k \geq j.$$

Returning now to (28) and (32), and comparing these results, it will be seen that there is a difference in the length of time that any demand, R_η, is accepted in the two sets under consideration. This difference in length of time is $(N_\eta' - N_\eta)$ and may be calculated directly from (28) and (32) in case $h \neq 1$, and from (29) and (33) in case $h = 1$, as follows

$$(41) \quad (N_\eta' - N_\eta) = (N_1' - c_\eta) - (N_1 - c_\eta) = N_1' - N_1, \qquad h \neq 1.$$

$$(42) \quad (N_\eta' - N_\eta) = (N_2' + c_2 - c_\eta) - (N_1 - c_\eta) = N_2' + c_2 - N_1, \qquad h = 1.$$

It will be noted that this difference in length of time is constant with respect to η.

In case $h \neq 1$ this constant difference, $(N_1' - N_1)$, may be evaluated by substituting the value of N_1' from (31). This gives

$$(N_1' - N_1) = \frac{A + \sum_1^k c_\eta R_\eta - c_h R_h - \sum_1^k N_1 R_\eta + N_1 R_h}{\sum_1^k R_\eta - R_h}$$

$$= \frac{A - \sum_1^k (N_1 - c_\eta) R_\eta + (N_1 - c_h) R_h}{\sum_1^k R_\eta - R_h}.$$

But $(N_1 - c_\eta) = N_\eta$ and $(N_1 - c_h) = N_h$, cf. (13), so that

$$(43) \qquad (N_1' - N_1) = \frac{A - \sum_1^k N_\eta R_\eta + N_h R_h}{\sum_1^k R_\eta - R_h}.$$

Now the N's appearing in the right hand member of (43) refer to the set $1, \cdots, j$, and are given by (28). They do not extend beyond N_j; that is, $N_{j+1}, \cdots, N_k$ are all excluded because w_j is the critical price for the set. [Note that $k \geq j$ by (40).] Therefore, in (43),

$$\sum_1^k N_\eta R_\eta = \sum_1^j N_\eta R_\eta = A,$$

and (43) becomes

$$(44) \quad N_\eta' - N_\eta = N_1' - N_1 = \frac{N_h R_h}{\displaystyle\sum_1^k R_\eta - R_h} = \frac{A_h}{\displaystyle\sum_1^k R_\eta - R_h} \equiv \epsilon.$$

By exactly similar reasoning, the same result is obtained when $h = 1$.

The effect, then, of arbitrarily rejecting the demand for land at the price w_h is to increase (because ϵ is obviously positive) the length of time that each of the non-rejected demands is accepted by the *same* length of time, ϵ, and this length of time is equal to the area which was allocated to the price w_h in the set $1, \cdots, j$, divided by the new total demand rate.

It may now be shown that $V_0(j)$, Eq. (27), is greater than $V_0'(k)$, Eq. (29). The proof is as follows:

(44) may be written

$$(45) \qquad N_h R_h = \epsilon \left(\sum_1^k R_\eta - R_h \right).$$

Multiplying both sides of this equation by w_h there results

$$(46) \qquad R_h w_h = \frac{1}{N_h} \epsilon \left(\sum_1^k R_\eta - R_h \right) w_h.$$

Since

$$a\daleth < \epsilon,$$

$$(47) \qquad R_h w_h > \frac{1}{N_h}\left[\sum_1^k R_\eta w_h a_{\overline{\epsilon}|} - R_h w_h a_{\overline{\epsilon}|} \right].$$

Both sides of (47) may be multiplied by $a_{\overline{N_h}|}$, to give

$$(48) \qquad R_h w_h a_{\overline{N_h}|} > \frac{a_{\overline{N_h}|}}{N_h}\left[\sum_1^k R_\eta w_\eta a_{\overline{\epsilon}|} - R_h w_h a_{\overline{\epsilon}|} \right].$$

But $u^{N_h} < \dfrac{a_{\overline{N_h}|}}{N_h}$, therefore,

$$(49) \qquad R_h w_h a_{\overline{N_h}|} > \sum_1 R_\eta w_h a_{\overline{\epsilon}|} u^{N_h} - R_h w_h a_{\overline{\epsilon}|} u^{N_h}.$$

From (10), $\dfrac{\log \dfrac{w_\eta}{w_1}}{\log u} = c_\eta$, and hence $w_\eta = u^{c_\eta} w_1$ and

$$w_h = u^{c_h} w_1 = u^{c_h} w_\eta u^{-c_\eta} = w_\eta u^{-(c_\eta - c_h)} = w_\eta u^{-N_h + N_\eta}.$$

Substituting this value of w_h in the right hand member of (49) gives

$$R_h w_h a_{\overline{N_h}|} > \sum_1^k R_\eta w_\eta a_{\overline{\epsilon}|} u^{N\eta} - R_h w_h a_{\overline{\epsilon}|} u^{N_h}, \text{ or}$$

$$(50) \quad R_h w_h a_{\overline{N_h}|} > \sum_1^h R_\eta w_\eta a_{\overline{\epsilon}|} u^{N\eta} - R_h w_h a_{\overline{\epsilon}|} u^{N_h} + \sum_{j+1}^k R_\eta w_\eta a_{\overline{\epsilon}|} u^{N\eta}.$$

Now ϵ is the length of time *after* N_η *years* that the demand in the price group w_η is accepted in the case in which group w_h is arbitrarily rejected. The initial value of these additional sales is $R_\eta w_\eta a_{\overline{\epsilon}|}$ deferred N_η years (or $R_\eta w_\eta a_{\overline{\epsilon}|} u^{N_h}$). In each of the price groups $w_1, \cdots, w_j$, exclusive of w_h, the demand is accepted for a period ϵ years *longer* than in the case in which w_h is included. The initial value of these additional sales in groups $w_1, \cdots, w_j$, exclusive of w_h, is

$$\left(\sum_1^j R_\eta w_\eta a_{\overline{\epsilon}|} u^{N\eta} - R_h w_h a_{\overline{\epsilon}|} u^{N_h} \right).$$

The remaining terms in the right hand member of (50) are $\sum_{j+1}^k R_\eta w_\eta a_{\overline{\epsilon}|} u^{N\eta}$, each term of which is of the form

$$R_{j+\delta} w_{j+\delta} a_{\overline{\epsilon}|} u^{N_{j+\delta}}.$$

The initial value produced by accepting the demand in groups $w_{j+1}, \cdots, w_k$, may be computed by noting that the length of time the demand in group $w_{j+\delta}$ is accepted is $N'_{j+\delta} = N_{j+\delta} + \epsilon$, from (44). $N_{j+\delta}$ is negative[2] because w_j is the critical price in the set $w_1, \cdots, w_\epsilon$. These demands $R_{j+1}, \cdots, R_k$, are accepted from the beginning, hence there is no deferment of the annuity. The initial value of $R_{j+\delta} w_{j+\delta}$ for $N'_{j+\delta}$ years is

$$R_{j+\delta} w_{j+\delta} a_{\overline{\epsilon - |N_{j+\delta}|}|} = R_{j+\delta} w_{j+\delta} \frac{a_{\overline{\epsilon}|} - a_{\overline{|N_{j+\delta}|}|}}{u^{|N_{j+\delta}|}}.$$

Since $N_{j+\delta}$ is negative,[2]

$$R_{j+\delta} w_{j+\delta} a_{\overline{\epsilon}|} u^{N_{j+\delta}} = R_{j+\delta} w_{j+\delta} \frac{a_{\overline{\epsilon}|}}{u^{|N_{j+\delta}|}}.$$

It is obvious that,

$$\frac{a_{\overline{\epsilon}|}}{u^{|N_{j+\delta}|}} > \frac{a_{\overline{\epsilon}|} - a_{\overline{|N_{j+\delta}|}|}}{u^{|N_{j+\delta}|}},$$

therefore,

$$(51) \quad R_{j+\delta} w_{j+\delta} a_{\overline{\epsilon - |N_{j+\delta}|}|} < R_{j+\delta} w_{j+\delta} a_{\overline{\epsilon}|} u^{N_{j+\delta}}.$$

[2] The argument still holds for the unique case $N_{j+1} = 0$.

From (51),

$$\text{(52)} \qquad \sum_{j+1}^{k} R_q w_q a_{\overline{e-|N_q|}} < \sum_{j+1}^{k} R_q w_q a_{\overline{e}} u^{N_q}.$$

Substituting (52) in (50) gives

$$\text{(53)} \quad R_h w_h a_{\overline{N_h}} > \left[\sum_{1}^{j} R_q w_q a_{\overline{e}} u^{N_q} - R_h w_h a_{\overline{e}} u^{N_h} \right] + \sum_{j+1}^{k} R_q w_q a_{\overline{e-|N_q|}}.$$

The left-hand member of (53) represents the initial value lost by excluding the price group w_h. The term in brackets in the right-hand member represents the *additional* initial value produced in the groups $w_1, \cdots, w_j$, exclusive of w_h, by rejecting the group w_h. The remaining term represents the initial value produced by introducing the additional lower-price groups $w_{j+1}, \cdots, w_k$. The expression (53) states that the value lost by excluding group w_h is greater than the value gained in the other groups. Therefore, $V_0(j) > V_0'(k)$.

The foregoing argument can be repeated by treating the $(k-h)$ set as an original set and arbitrarily exluding a group w_i which will change the critical price from w_k to, say, w_l. Denoting the initial value of this set by $V_0''(l)$, it follows that $V_0'(k) > V_0''(l)$, and so on, so that

$$\text{(54)} \qquad V_0(j) > V_0'(k) > V_0''(l) > \cdots > V_0'' \cdots '(s),$$

where the number of prime ($'$) marks indicate the number of groups arbitrarily rejected from the original set $1, \cdots, s$. Therefore,

$$\text{(55)} \quad \left\{ \begin{array}{l} V_0(j) = \displaystyle\sum_{1}^{j} R_q w_q a_{\overline{N_q}} \\[2mm] \text{where} \quad \displaystyle\sum_{1}^{j} N_q R_q = A \\[2mm] \text{and} \quad N_q = \dfrac{A + \displaystyle\sum_{1}^{j} c_q R_q}{\displaystyle\sum_{1}^{j} R_q} - c_q > 0, \end{array} \right.$$

is the *practical maximum value* of area A.

The following theorem may now be stated:

THEOREM. The owner of an area of land A for which there is a series of continuous and constant demands on the part of prospective purchasers at the rates $R_1, \cdots, R_s$, (expressed in area units per year) and at the corresponding unit prices $w_1, \cdots, w_s$, where the prices are so

numbered by subscripts that $w_1 > w_2 > w_3 > \cdots > w_s$, and with the unsold portion of area A subject to an annual tax at a constant tax rate ρ levied on the actual value at the beginning of each year and paid one year later and with no incidental income from the unsold portions during the periods prior to the sale of such portions, will realize the maximum value by the sale of the property according to a specific schedule as follows: For each of the price groups w_η ($\eta = 1, \cdots, j$), he should allocate an area A_η to be sold at the rate R_η for N_η years, where $R_\eta N_\eta = A_\eta$

$$\text{and } N_\eta = \frac{A + \sum_1^j c_\eta R_\eta}{\sum_1^j R_\eta}, \text{ in which expression } c_\eta = \frac{\log \dfrac{w_\eta}{w_1}}{\log u} \text{ and } u = \frac{1}{1 + i + \rho},$$

i being the rate of return on the value and j being the subscript of that particular price w_j, called the *critical price*, which is the *lowest* w in the set $w_1, \cdots, w_s$, which when included in the summations in the equation for N_η gives a *positive* value of N.

COROLLARY I. Under the conditions stated in the Theorem above, the owner of area A should reject all demands in price groups lower than the critical price w_j, and should accept the total annual demand in price group w_j for a period of N_j years and reject this demand entirely after N_j years. Similarly, he should accept the total annual demand in price group w_{j-1} for a period of N_{j-1} years and reject it entirely after N_{j-1} years, and so on for all the higher price groups.

COROLLARY II. Under the conditions stated in the above Theorem, the owner of area A should accept the total annual demand, no matter how small it may be, in a *higher* price group instead of rejecting it in favor of the total annual demand in a *lower* price group, no matter how large such demand may be; subject to the limitation that the demand in any group should not be accepted for more than the number of years calculated by the expression for N_η given in the Theorem above.

COROLLARY III. Under the conditions stated in the Theorem above, the lower the price at which a demand for land appears, the shorter will be the period of time during which that demand should be accepted, regardless of how great the demand may be.

III. FILLING OF REJECTED DEMANDS IN LESS DESIRABLE LOCATIONS

In the Theorem and its corollaries stated in Part II, it was shown that the owner of an area A should reject certain opportunities to sell land in order to realize the maximum value in the sale of his land al-

though during the period of selling he still has a supply of vacant[3] land for sale.

Now it may be assumed that these rejected demands for lower price lands are filled somewhere outside of area A. Assume another area of land, B, different from A, also in single ownership, so situated that the owner of B may, if he wishes, accept the demands rejected by the owner of A.

There will be no demand on the owner of area B for land in the price group w_j until after N_j years, because the owner of A will absorb the entire demand at this price for N_j years, and similarly for the other groups in the set $w_1, \cdots, w_j$, provided that prospective purchasers prefer land in area A to land in area B and take land in B as second choice. The owner of area B will have, however, immediate and continuing demands for land in the price groups $w_{j+1}, \cdots, w_s$, inasmuch as these groups are rejected entirely by the owner of A.

If $n_1 =$ the number of years that the price group w_1 is accepted by the owner of B and, in general, if $n_\eta =$ the number of years that the price group w_η is accepted by the owner of B, w_η being any one of the set $w_1, \cdots, w_j$, the value equation for area B may be written (cf. (2)),

$$(56) \quad \begin{aligned} V_0 = & R_{j+1}w_{j+1}a_{\overline{N_{j+1}}|} + R_{j+2}w_{j+2}a_{\overline{N_{j+2}}|} + \cdots + R_s w_s a_{\overline{N_s}|} \\ & + R_1 w_1 a_{\overline{n_1}|}u^{N_1} + R_2 w_2 a_{\overline{n_2}|}u^{N_2} + \cdots + R_j w_j a_{\overline{n_j}|}u^N, \end{aligned}$$

subject to the condition that

$$(57) \quad \begin{aligned} R_{j+1}N_{j+1} + R_{j+2}N_{j+2} + \cdots + R_s N_s + R_1 n_1 \\ + R_2 n_2 + \cdots + R_j N_j = B. \end{aligned}$$

The factors $u^{N_1}, \cdots, u^{N_j}$, appear in the value equation because the demands in the groups $w_1, \cdots, w_j$, do not appear in area B until after the lapse of $N_1, \cdots, N_j$, years, respectively, and hence each annuity of the form $R_\eta w_\eta$ is deferred for N_η years.

The value of area B as given by (56) and (57) will, obviously, depend on the values of $N_{j+1}, \cdots, N_s$, and $n_1, \cdots, n_j$, which are selected. To determine the maximum value of V_0 under the stated conditions, the procedure is the same as before. The value equation may be written

$$(58) \quad \begin{aligned} V_0 = \frac{1}{i+\rho} & [R_{j+1}w_{j+1}(1-u^{N_{j+1}}) + R_{j+2}w_{j+2}(1-u^{N_{j+2}}) + \cdots \\ & + R_s w_s(1-u^{N_s}) + R_1 w_1(1-u^{n_1})u^{N_1} \\ & + R_2 w_2(1-u^{n_2})u^{N_2} + \cdots + R_j w_j(1-u^{n_j})u^{N_j}], \end{aligned}$$

[3] Since in setting up the value equation (2) it was tacitly assumed that there was no incidental income to the owner from the utilization by him of the unsold portions, the unsold land, at any time, may be considered as vacant.

and the total differential is

$$
\begin{aligned}
dV_0 = -\frac{\log u}{i + \rho}\big[& R_{j+1}w_{j+1}u^{N_{j+1}}dN_{j+1} + R_{j+2}w_{j+2}u^{N_{j+2}}dN_{j+2} \\
& + \cdots + R_s w_s u^{N_s}dN_s + R_1 w_1 u^{N_1}u^{n_1}dn_1 \\
& + R_2 w_2 u^{N_2}u^{n_2}dn_2 + \cdots + R_j w_j u^{N_j}u^{n_j}dn_j\big].
\end{aligned}
$$
(59)

The total differential of B in (57) is zero because B is constant by hypothesis, therefore,

$$
\begin{aligned}
dB = & R_{j+1}dN_{j+1} + R_{j+2}dN_{j+2} + \cdots + R_s dN_s \\
& + R_1 dn_1 + R_2 dn_2 + \cdots + R_j dn_j = 0.
\end{aligned}
$$
(60)

The value of dN_{j+1}, given by (60), substituted in (59) gives

$$
\begin{aligned}
dV_0 = \frac{\log u}{i + \rho}\big[& R_{j+2}(w_{j+1}u^{N_{j+1}} - w_{j+2}u^{N_{j+2}})dN_{j+2} \\
& + R_{j+3}(w_{j+1}u^{N_{j+1}} - w_{j+3}u^{N_{j+3}})dN_{j+3} + \cdots \\
& + R_s(w_{j+1}u^{N_{j+1}} - w_s u^{N_s})dN_s \\
& + R_1(w_{j+1}u^{N_{j+1}} - w_1 u^{N_1}u^{n_1})dn_1 \\
& + R_2(w_{j+1}u^{N_{j+1}} - w_2 u^{N_2}u^{n_2})dn_2 + \cdots \\
& + R_j(w_{j+1}u^{N_{j+1}} - w_j u^{N_j}u^{n_j})dn_j\big].
\end{aligned}
$$
(61)

For the maximum value of V_0, it is necessary that each of the differential coefficients be zero, as before, hence,

$$
\begin{cases}
\log w_{j+1} + N_{j+1}\log u = \log w_{j+2} + N_{j+2}\log u \\
\qquad\qquad\qquad\quad = \log w_{j+3} + N_{j+3}\log u \\
\qquad\cdots\cdots\cdots\cdots\cdots \\
\qquad\qquad\qquad\quad = \log w_s + N_s\log u
\end{cases}
$$
(62)

and

$$
\begin{cases}
\log w_{j+1} + N_{j+1}\log u = \log w_1 + (N_1 + n_1)\log u \\
\qquad\qquad\qquad\quad = \log w_2 + (N_2 + n_2)\log u \\
\qquad\cdots\cdots\cdots\cdots\cdots \\
\qquad\qquad\qquad\quad = \log w_j + (N_j + n_j)\log u.
\end{cases}
$$
(63)

The first equation in (62) gives

$$N_{j+1} - N_{j+2} = \frac{\log w_{j+2} - \log w_{j+1}}{\log u}$$

$$= \frac{\log \dfrac{w_{j+2}}{w_1} - \log \dfrac{w_{j+1}}{w_1}}{\log u} = c_{j+2} - c_{j+1},$$

and, by the same process, the values of $(N_{j+1} - N_{j+3})$ etc., may be determined, hence,

$$(64) \qquad \begin{cases} N_{j+1} - N_{j+2} = c_{j+2} - c_{j+1}, \\ N_{j+1} - N_{j+3} = c_{j+3} - c_{j+1}, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ N_{j+1} - N_s = c_s - c_{j+1}. \end{cases}$$

The same procedure applied to (63) gives

$$(65) \qquad \begin{cases} N_1 + n_1 - N_{j+1} = c_{j+1}, \\ N_2 + n_2 - N_{j+1} = c_{j+1} - c_2, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ N_j + n_j - N_{j+1} = c_{j+1} - c_j. \end{cases}$$

Subtracting the second equation in (65) from the first gives

$$N_1 - N_2 + n_1 - n_2 = c_2, \quad \text{but} \quad N_1 - N_2 = c_2 \quad \text{by (10)}$$

and therefore $n_1 = n_2$. Subtracting the successive equations from the first produces the result

$$(66) \qquad n_1 = n_2 = n_3 = \cdots = n_j \equiv n.$$

This is an important result inasmuch as it states that the length of time the demands in price groups w_1 to w_j are to be accepted by the owner of area B are all identical regardless of what the demand rates may be.

By means of (66), (57) may be written

$$(67) \qquad R_{j+1}N_{j+1} + R_{j+2}N_{j+2} + \cdots + R_s N_s + n \sum_1^j R_q = B.$$

From the first equation in (65)

$$(68) \qquad n = N_{j+1} + c_{j+1} - N_1.$$

Substituting this value of n in (67) gives

$$R_{j+1}N_{j+1} + R_{j+2}N_{j+2} + \cdots + R_sN_s + N_{j+1}\sum_1^j R_\eta,$$
$$= B + (N_1 - c_{j+1})\sum_1^j R_\eta,$$

and, by combination of terms in N_{j+1},

$$\left(\sum_1^{j+1} R_\eta\right)N_{j+1} + R_{j+2}N_{j+2} + \cdots + R_sN_s$$

(69)

$$= B + (N_1 - c_{j+1})\sum_1^j R_\eta.$$

This equation and the equations (64) determine the values of $N_{j+1}, \cdots, N_s$.

(70)
$$\begin{cases}
\left(\sum_1^{j+1} R_\eta\right)N_{j+1} + R_{j+2}N_{j+2} + \cdots + R_sN_s = B + (N_1 - c_{j+1})\sum_1^j R_\eta. \\[2mm]
N_{j+1} \quad -N_{j+2} \qquad\qquad = c_{j+2} - c_{j+1}. \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
N_{j+1} \qquad \cdots -N_{j+\eta} \qquad = c_{j+\eta} - c_{j+1}. \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
N_{j+1} \qquad\qquad\qquad -N_s = c_s - c_{j+1}.
\end{cases}$$

The solution may be accomplished by multiplying the first equation by unity, the second equation by R_{j+2}, the third by R_{j+3}, and so on to the last, which is multiplied by R_s, and adding the resulting equations. Hence,

$$\left(\sum_1^s R_\eta\right)N_{j+1} = B + (N_1 - c_{j+1})\sum_1^j R_\eta + \sum_2^{s-j}(c_{j+\eta} - c_{j+1})R_{j+\eta}$$

(71)

$$= B + N_1\sum_1^j R_\eta + \sum_{j+1}^s c_\eta R_\eta - c_{j+1}\left(\sum_1^j R_\eta + \sum_{j+1}^s R_\eta\right).$$

Now, $N_1\sum_1^j R_\eta = A + \sum_1^j c_\eta R_\eta$, by (12), (the upper limit being the subscript of the critical price for area A, which is j); substituting this value in (71) and combining terms gives

$$N_{j+1} = \frac{1}{\sum_1^s R_\eta}\left[A + B + \sum_1^s c_\eta R_\eta - c_{j+1}\sum_1^s R_\eta\right]$$

(72)

$$= \frac{A + B + \sum_1^s c_\eta R_\eta}{\sum_1^s R_\eta} - c_{j+1}.$$

The general term, $N_{j+\eta}$, is given by (64),

$$(73) \qquad N_{j+\eta} = N_{j+1} - (c_{j+\eta} - c_{j+1}) = \frac{A + B + \sum_1^s c_\eta R_\eta}{\sum_1^s R_\eta} - c_{j+\eta}.$$

The allocation of area B into parts to be sold at the various available prices $w_1, \cdots, w_s$, so as to produce the maximum value of area B is accomplished by multiplying the demand rate in each price group by its appropriate N or by n, as the case may be. Thus

$$(74) \text{ for } 1 \geqq \eta \geqq j, B_\eta = R_\eta \left[\frac{A + B + \sum_1^s c_\eta R_\eta}{\sum_1^s R_\eta} - N_1 \right] \text{Cf. (68) \& (72).}$$

$$(75) \quad \text{for } j + 1 \geqq \eta \geqq s, B_\eta = R_\eta \left[\frac{A + B + \sum_1^s c_\eta R_\eta}{\sum_1^s R_\eta} - c_{j+\eta} \right].$$

As in the case of area A, area B, too, has a critical price under the assumed set of conditions. For a practical maximum value, the upper limit in the summations in (73), (74), and (75), cannot be greater than the subscript of this critical price for B.

IV. MULTIPLE OWNERSHIP OF LAND

In the preceding sections, it has been assumed that there exist certain constant and continuous demands for land at certain unit prices and that the first choice of location on the part of prospective purchasers is within area A and that their second choice is within area B. It was assumed that area A was initially in one ownership and area B was initially in another ownership and that each so sells off his land as to produce the maximum value for himself. The owner of area A so apportions his land among the several price and demand groups as to set aside an area $A_\eta = R_\eta(N_1 - c_\eta)$, Eq. (14), for each group η. At the same time, the owner of area B so apportions his land among the several price and demand groups as to set aside an area B_η, where

$$B_\eta = R_\eta(X - N_1) \text{ for } 1 \geqq \eta \geqq j, \text{ or } B_\eta = R_\eta(X - c_{j+\eta}) \text{ for } j + 1 \geqq \eta \geqq k,$$

in which expressions, $k =$ subscript of critical price for area B and

$$X \equiv \frac{A + B + \sum_{1}^{k} c_{\eta} R_{\eta}}{\sum_{1}^{k} R_{\eta}}.$$

The following table shows the allocation of area in A and B separately, for each price group, and the sum of the areas in A and B allocated to each price group.

Subscript of price group	Allocation of areas		Total
	to area A	to area B	
1	$R_1 N_1$	$R_1 n \quad = R_1(X - N_1)$	$R_1 X$
2	$R_2(N_1 - c_2)$	$R_2 n \quad = R_2(X - N_1)$	$R_2(X - c_2)$
3	$R_3(N_1 - c_3)$	$R_3 n \quad = R_3(X - N_1)$	$R_3(X - c_3)$
. . .	. . . .	. . . .	. . . .
j	$R_j(N_1 - c_j)$	$R_j n \quad = R_j(X - N_1)$	$R_j(X - c_j)$
$j+1$		$R_{j+1} N_{j+1} = R_{j+1}(X - c_{j+1})$	$R_{j+1}(X - c_{j+1})$
$j+2$		$R_{j+2} N_{j+2} = R_{j+2}(X - c_{j+2})$	$R_{j+2}(X - c_{j+2})$
. . .		. . . .	. . . .
k		$R_k N_k \quad = R_k(X - c_k)$	$R_k(X - c_k)$

$$\text{Total} = \left| \sum_{1}^{k} R_{\eta}(X - c_{\eta}) = A + B \right.$$

Consider now the case in which the areas A and B are in the *same* instead of different and *competing* ownership. The owner of this combined area, $(A+B)$, to produce the maximum value, will allocate the total area among the several price groups, $w_1, \cdots, w_k$, by means of (14). The result is obtained by substituting $(A+B)$ for A in (14),

$$(76) \quad (A + B)_{\eta} = R_{\eta} \left[\frac{A + B + \sum_{1}^{k} c_{\eta} R_{\eta}}{\sum_{1}^{k} R_k} - c_{\eta} \right] = R_{\eta}(X - c_{\eta})$$

and by comparison with the quantities in the fourth column of the pre-

ceding table, it will be seen that the allocation of the total area $(A+B)$ to the several price groups is precisely the same as the allocation in the case in which A and B are in separate and competing ownership.

The area B has its critical price and all demands for B land at prices lower than this critical price will be rejected to less desirable locations with an area C, and so on.

There are no limitations on the sizes of the areas A, B, C, D, etc., and these areas do not each have to be in one piece. Furthermore, the conclusion reached, namely, that the allocation of the entire area $A+B+C+D+ \cdots$, to the whole category of price groups is precisely the same as the allocation in the case in which each aggregate, A, B, C, etc., is in separate and competing ownership, removes the original limitations placed on the theory by the assumption of *monopolistic* ownership of each of the aggregates. The results are the same regardless of how many owners there are in each of the groups A, B, C, etc. It should be noted, however, that, as a practical matter, it is not the *true* future demand for land which controls the allocation and determines the land utilization and the dispersion of building development in the cities. In the nature of things, these true future demands can only be approximated and, in the absence of statistical studies, little is known about them at the present time. The result is that, in multiple ownership, there is a heterogeneous opinion on the part of land owners as to future demand so that city development is not the same as it would be were accurate demand predictions available.

Los Angeles,
  California

PERIODOGRAM ANALYSIS WITH SPECIAL APPLICATION TO BUSINESS FAILURES IN THE UNITED STATES, 1867–1932[1]

By Benjamin Greenstein

I. INTRODUCTION

In investigating the periodicities of business failures, I am not assuming that the actual fluctuations of the data are chiefly of a periodic character, nor even that there are periodicities in the observations. I am merely supposing that the presence of periodic elements in the given fluctuations is possible. It is my object to isolate those elements, if they are present, and to find out whether their action is continuous or intermittent.

Attempts at the mathematical definition of the periods in economic cycles are not new.[2] The methods of periodogram analysis, Fourier sequence, and fitting of Fourier series, have been used for that purpose. Whether or not these refined mathematical methods are suited to the study of economic series is a debated subject. Professor Moore is quite satisfied that he has found by these methods an eight-year cycle in several economic series.[3] W. L. Crum, on the other hand, is dissatisfied with the results of periodogram analysis in his study of the cycles in the rates on commercial paper. One of his most important difficulties and his conclusions regarding it are in the following statement:

> The length of the period in an economic series may, in general, well be variable. In the present case, our data do not show conclusively that it is. On the other hand it is not possible to conclude from our results that it is not. The actual fit of the theoretical periodic curves to our data is not good, and we cannot be sure whether the lack of fit is due merely to chance irregular deviations or to a real modification of the period itself. In the absence of conclusive proof to the contrary, *we believe that the economic period should not be assumed constant*. If,

[2] The following studies of that nature may be mentioned: H. L. Moore, *Economic Cycles; their Law and Cause* (1914); H. L. Moore, *Generating Economic Cycles* (1923); W. L. Crum, "Cycles of Rates on Commercial Paper," *Review of Economic Statistics*, 1923, pp. 17–27.

[3] H. L. Moore, *Generating Economic Cycles*, Chap. III–IV.

therefore, dropping the notion of a constant economic period, we understand by
the term "period" the average length of the cycles, the periodogram can assist
in finding the average length, first, if it is fairly typical—that is, if the actual
cycles do not deviate greatly from it—and, second, if the form of the cycle does
not undergo great change. . . . [4]

As annual data are used in the present study, no disturbing seasonal
variation is involved. No attempt is here made to get theoretical values
of the variable for time intervals shorter than the year. Finally, Crum's
belief that the economic cycle should not be assumed constant has
not been ignored.[5] Thus, most of Crum's criticisms of the application
of the periodogram to economic series have been met. Of course, there
still remains the disturbing effect of irregularities which, to the detri-
ment of an analysis of periodicities, is characteristic of economic data.

II. THE DATA

An investigation of business failures in the United States has to
face the fact that the required data as given in *Dun's Review* differ
from those in *Bradstreet's Weekly*. Not only do we have two conflicting
reports but each of these is not altogether consistent with itself. In
Table I, the percentage ratio of failures for each year was obtained by
dividing the number of failures by the number of concerns. The sources
of the numbers of failures and of concerns are indicated. If the reader
should compare my ratios with those given in the quoted sources, he
would find numerous discrepancies.[6] Many of the ratios which they
give are not the actual quotients of the required divisions but appar-
ently the results of incorrect division.[7]

[4] W. L. Crum, *op. cit.*, p. 24.

[5] It is noteworthy that the methods of the periodogram and Fourier sequence
were developed in the investigation of observations on sunspots, which have a
variable period.

[6] The ratios that I derive from Dun's data for 1868, 1875, 1879, 1880, 1881,
1884, 1887, 1888, 1892, 1900, 1905, 1906, 1909, and 1927, differ from those
given in *Dun's Review*, Jan. 21, 1933, p. 10. The percentage ratios given in
Bradstreet's Business Year Review, Jan. 28, 1933, p. 171, are in 23 instances out
of 52 smaller by .01 than those obtained by correct division. Bradstreet's com-
puters have apparently rarely taken the trouble to carry their division to one
digit beyond the last one given by them, and to raise the latter by 1 when it is
followed by a digit greater than 5.

[7] Although there are several differences between the percentages of commercial
failures in *Dun's Review*, Jan. 11, 1930, p. 7, and the corresponding ratios in
Dun's Review, Jan. 8, 1927, p. 7, the actual numbers of failures given in the two
issues are in perfect agreement. It, therefore, seems likely that the discrepancies
are due to errors in division. Actual division of the numbers of failures by the
numbers of concerns reveals that the earlier issue is correct in some of the in-
stances and the later in others, and that in one case both are wrong. In a reply
to a letter of inquiry regarding the discrepancies, which was kindly sent on my
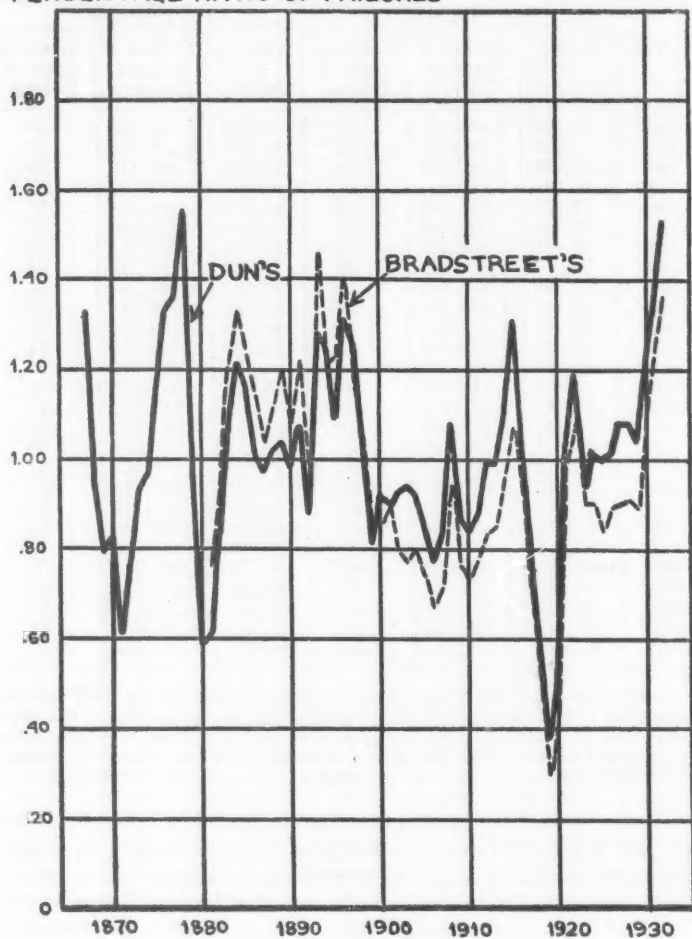behalf by Professor John H. Cover, *Dun's Review* asked for a statement of the

TABLE I

THE PERCENTAGE RATIOS OF BUSINESS FAILURES TO THE TOTAL NUMBER OF BUSINESS CONCERNS IN THE UNITED STATES, 1867–1932*

Year	Dun's			Bradstreet's		
	Number of Failures	Number of Business Concerns	Percentage Ratio of Failures	Number of Failures	Number of Business Concerns	Percentage Ratio of Failures
1867	2,780	209,720	1.33			
1868	2,608	278,840	.94			
1869	2,799	352,674	.79			
1870	3,546	427,230	.83			
1871	2,915	475,145	.61			
1872	4,069	528,970	.77			
1873	5,183	559,764	.93			
1874	5,830	600,490	.97			
1875	7,740	642,420	1.20			
1876	9,092	681,900	1.33			
1877	8,872	652,006	1.36			
1878	10,478	674,741	1.55			
1879	6,658	702,157	.95			
1880	4,375	746,823	.59			
1881	4,735	781,689	.61	5,929	780,000	.76
1882	6,788	822,256	.83	7,635	820,000	.93
1883	9,184	863,993	1.06	10,299	855,000	1.20
1884	10,968	904,759	1.21	11,620	875,000	1.33
1885	10,637	919,990	1.16	11,116	890,000	1.25
1886	9,824	969,841	1.01	10,568	920,000	1.15
1887	9,634	994,281	.97	9,740	933,000	1.04
1888	10,679	1,046,662	1.02	10,587	955,000	1.11
1889	10,882	1,051,140	1.04	11,719	978,000	1.20
1890	10,907	1,110,590	.98	10,673	989,420	1.08
1891	12,273	1,142,951	1.07	12,394	1,018,021	1.22
1892	10,344	1,172,705	.88	10,270	1,035,564	.99
1893	15,242	1,193,113	1.28	15,508	1,059,014	1.46
1894	13,885	1,114,174	1.25	12,724	1,047,974	1.21
1895	13,197	1,209,282	1.09	12,958	1,053,633	1.23
1896	15,088	1,151,579	1.31	15,094	1,079,070	1.40
1897	13,351	1,058,521	1.26	13,083	1,086,056	1.20
1898	12,186	1,105,830	1.10	11,615	1,093,373	1.06
1899	9,337	1,147,595	.81	9,642	1,125,873	.86
1900	10,774	1,174,300	.92	9,912	1,161,639	.85
1901	11,002	1,219,242	.90	10,648	1,201,862	.89
1902	11,615	1,253,172	.93	9,973	1,238,973	.80
1903	12,069	1,281,481	.94	9,775	1,272,909	.77
1904	12,199	1,320,172	.92	10,417	1,307,746	.80
1905	11,520	1,357,455	.85	9,967	1,352,947	.74

* The numbers of failures and of concerns are taken from *Dun's Review*, Jan. 21, 1933, p. 10, and *Bradstreet's Weekly* "Business Year Review," Jan. 28, 1933, p. 171.

specific discrepancies. When its request was fulfilled, there was no reply. It is interesting to note that Jan. 21, 1933, over two years after this correspondence, the erroneous ratios were still reprinted.

TABLE I (*Continued*)

Year	Dun's			Bradstreet's		
	Number of Failures	Number of Business Concerns	Percentage Ratio of Failures	Number of Failures	Number of Business Concerns	Percentage Ratio of Failures
1906	10,682	1,392,949	.77	9,385	1,401,085	.67
1907	11,725	1,418,075	.83	10,265	1,447,680	.71
1908	15,690†	1,447,554	1.08	14,044	1,487,813	.94
1909	12,924	1,486,389	.87	11,845	1,543,444	.77
1910	12,652	1,515,143	.84	11,573	1,592,509	.73
1911	13,441	1,525,024	.88	12,646	1,637,650	.77
1912	15,452	1,564,279	.99	13,812	1,673,452	.83
1913	16,037	1,616,517	.99	14,551	1,718,345	.85
1914	18,280	1,655,496	1.10	16,769	1,749,101	.96
1915	22,156	1,674,788	1.32	19,035	1,770,914	1.07
1916	16,993	1,707,639	1.00	16,496	1,790,776	.92
1917	13,855	1,733,225	.80	13,029	1,828,464	.71
1918	9,982	1,708,061	.58	9,331	1,824,104	.51
1919	6,451	1,710,909	.38	5,515	1,843,066	.30
1920	8,881	1,821,409	.49	8,463	1,958,042	.43
1921	19,652	1,927,304	1.02	20,014	2,049,323	.98
1922	23,676	1,983,106	1.19	22,415	2,074,617	1.08
1923	18,718	1,996,004	.94	19,159	2,136,921	.90
1924	20,615	2,047,302	1.01	19,712	2,195,626	.90
1925	21,214	2,113,300	1.00	18,859	2,242,317	.84
1926	21,773	2,158,400	1.01	20,024	2,258,423	.89
1927	23,146	2,171,700	1.07	20,267	2,255,429	.90
1928	23,842	2,199,000	1.08	20,373	2,246,208	.91
1929	22,909	2,212,779	1.04	19,703	2,213,935	.89
1930	26,355	2,183,008	1.21	24,107	2,185,358	1.10
1931	28,285	2,125,288	1.33	26,381	2,125,675	1.24
1932	31,822	2,076,580	1.53	28,773	2,122,845	1.36

† This number is quoted as 15,590 in *Dun's Review*, Jan. 21, 1933. As the quotation is inconsistent both with the adjoining percentage ratio in the same issue and with the number of failures as given in earlier issues, it is apparently erroneous. I have examined the Annual Summary Numbers of *Dun's Review* for 1921–1932 and found the number in question given as 15,690 in all of them.

Dun's data of failures have been used by both Mitchell and Snyder in their studies of business cycles; also Bradstreet's data have been presented by the former. Neither writer has eliminated the errors of of division included in the sources.[8] Indeed, if I should judge by the issues of *Dun's Review* which I have used, Mitchell even introduces an additional error. Dun's percentage ratio of failures for 1903, as given by him, is not only erroneous but also different from that of

[8] Compare Table 125 in Mitchell's *Business Cycles* (1913) with my Table I and Chart 49 in Snyder's *Business Cycles and Business Measurements* with my Figure 1. In Snyder's chart, note especially the ordinates that represent the ratios of failures for 1909–1911.

PERCENTAGE RATIO OF FAILURES

FIGURE 1.—The percentage ratios of business failures to the total number
of business concerns in the United States, 1867–1932.

Dun.[9] Possibly, however, the copy of *Dun's Review* which he used contained this additional error.

Snyder does not discuss the reliability of the data of failures which he uses. He does not even mention that Bradstreet's reports differ from Dun's. Mitchell, however, who uses both sources, remarks on these differences:

The cause of these discrepancies—apart from the difficulty of securing accurate information—is not altogether clear. Bradstreet's explains that its figures do not include failures of banks, professional men, farmers, stock-brokers, or real estate dealers, and that they do not include cases of "failure to succeed" which involve no loss to creditors. Whether Dun's figures do include these classes is not known, because the figures are published without explanation of their scope.[10]

In Figure 1, the ratios of failures which are derived from Dun's data are compared with those obtained from Bradstreet's. Dun's ratios are generally the lower of the two before 1897, but always the higher from 1900 on. While the absolute differences between the two are often considerable, the general movements of the series under comparison are very similar. The timing of peaks and depressions is generally the same in both. It is, therefore, reasonable to expect that during their common time interval, 1881–1932, the periods found in either will approximate in length those of the other.

To increase our confidence in the significance of Dun's ratios, we have the following statement by Snyder:

An index of business failures becomes an inverse measure of business cycles. This is shown in Chart 49, where the percentages of firms failing to the total number in business each year is plotted against the annual averages of the Clearings Index of Business. There is a high degree of negative correlation between the two series. The Clearings Index registers prosperity by a high level, the business failures index registers prosperity by a low level. The two series usually cross the "normal" line at the same time, and the one reaches its maximum simultaneously as the other reaches its minimum.[11]

[9] I have noted before that Dun gives two variants of its ratios of failures, which disagree in several instances. One is called "The Percentages of Commercial Failures to the Total Number of Business Concerns in the United States" and the other, "The Ratio of Commercial Failures to Each 10,000 Business Concerns." Examples of the former are in *Dun's Review*, Jan. 11, 1930, p. 7 and *ibid.*, Jan. 16, 1932, p. 6; of the latter, *ibid.*, Jan. 8, 1927, p. 7 and Jan. 21, 1933, p. 10. Mitchell's variant is, except for one item, identical with Dun's for 1930. The percentage ratio for 1903, in which the two differ, is erroneously given by Mitchell as 1.12 and correctly by Dun as .94.

[10] Wesley C. Mitchell, *Business Cycles* (1913), p. 438.

[11] Carl Snyder, *Business Cycles and Business Measurements* (1927), p. 183. Most of the errors in Dun's ratios, which Snyder is using, are small. For the purpose of drawing the general conclusions, which I have quoted, the differences between his graph of Dun's ratios and mine are immaterial.

For the present study Dun's ratios are preferable to Bradstreet's, because they are reported for a greater number of years. Even the series composed of the former is unusually short for a study of periodicities. It is, however, the longest series at our disposal, and must of necessity be chosen.

III. THE METHODS OF ANALYSIS

A simple arithmetical method for investigating hidden periodicities is the "oscillation" method.[12] As it will not be used here, in spite of its convenience and simplicity, it is, perhaps, best to explain why. Yule writes of it:

The process may, in my very limited experience, lead to quite misleading results. Not only does the four-year grouping contain the two-year as well as the four-year amplitude, the six-year grouping contain the two-year and three-year as well as the six-year amplitude, and so on, but there may also be casual fluctuations. In one of my cases, for example, the difference between the maximum and minimum in the column-averages for determining a six-year period was 124.8, which would correspond to an amplitude of 62.4 on the above approximation.[13] The true amplitude was only 3.14. In another case the differences between the maximum and minimum in the column-averages for determining a five-year period was 222.8, which would correspond to an amplitude of 111.4 on the above approximation. The true amplitude was only 4.67.[14]

In view of Yule's warning, the following statement of Albert Eagle with respect to the "oscillation" method is surprising. He writes, "the theory of the problem that we have given . . . shows that more elaborate methods are not worth the extra labour of applying, and are

[12] A summary of Brunt's description of it may be helpful. Let $x_0, x_1, x_2, \cdots,$ be a series of observations taken at equal intervals of time α. To examine the observations for a possible period $p\alpha$, where p is an integer, the x's are written in n rows of p each. In the first column are the observations, $x_0, x_p, \cdots, x_{(n-1)p}$; in the second, $x_1, x_{p+1}, \cdots, x_{(n-1)p+1}$; in the last, $x_{p-1}, x_{2p-1}, \cdots, x_{np-1}$. The sums of the columns may be called $V_0, V_1, \cdots, V_{p-1}$, respectively. The difference between the largest of these sums and the smallest one "is a rough measure of the amplitude of the period of p intervals." This difference, or "oscillation," is computed for many trial periods. It is made the ordinate of a graph having the trial period as abscissa. "The resulting curve may be called the 'curve of periods.' It shows high peaks at the points at which real periodicities exist, and lower peaks at points corresponding to doubtful periodicities." David Brunt, *Combination of Observations* (1917), §95.

[13] Yule refers here to a quotation from Carse and Shearer, *A Course in Fourier's Analysis and Periodogram Analysis*, p. 34. They write that the "oscillation" of the sums of the columns is twice their amplitude if the series of sums "represents a purely harmonic variation, while if this is not the case it is a sufficiently accurate measure of the importance of the trial period in question." (Note is mine.)

[14] G. U. Yule, "On the Time Correlation Problem," *Journal of the Royal Statistical Society*, July, 1921, p. 525.

no more necessary than is the ideal diffraction grating, which only gives a single order spectrum, to the experimenter in spectroscopy.[15] My ignorance of spectroscopy forbids me to say how necessary is the ideal diffraction grating, which gives a single order spectrum. In harmonic analysis, however, I have considered more elaborate methods than the "oscillation" method necessary for the isolation of true periods. Eagle was probably unacquainted with Yule's criticism. Otherwise, he would certainly have considered the warning of this eminent statistician worthy of comment and would, perhaps, also have reconsidered his statement about the superfluity of elaborate methods.

One of the most important contributors to the theory and practice of the investigation of periodicities was Professor Schuster. He proposed the use of the periodogram, which is a graph of the squares of the amplitudes for a large number of trial periods. The square of the amplitude for a trial period p in pn observations, x_r, taken at equal intervals of time, is

$$R^2 = A^2 + B^2,$$

where, for integral values of n,

$$A = \frac{2}{pn} \sum_{r=0}^{pn-1} x_r \cos \frac{2\pi r}{p}, \text{ and}$$

$$B = \frac{2}{pn} \sum_{r=0}^{pn-1} x_r \sin \frac{2\pi r}{p}.$$

When pn, the number of observations used for obtaining the various amplitudes, varies considerably, the ordinate of the periodogram should be

$$S = R^2 pn.$$

"If a trial period . . . should fall near one of the actual periods of the observations, the resulting amplitude R . . . will be considerably greater than if the trial period . . . were considerably removed from the actual periods."[16] The approximate location of the actual periods may, therefore, be inferred from a study of the maxima of the periodogram.

As it is not possible to find the values of S for all conceivable trial periods, a limit to the closeness of two such periods to each other is necessary. Schuster gets such a limit from the fact that "any two periods which may be investigated should be independent, and two

[15] Albert Eagle, *A Practical Treatise on Fourier's Theorem and Harmonic Analysis* (1925), p. 178.

[16] Brunt, *Combination of Observations*, §89. The proof of the quoted statement is in §90. The various definitions of S are in §§93–94.

near periods will begin to be independent, when there is a final difference of phase of a quarter-period."[17] The condition for independence may be expressed algebraically in the form,

$$n(T - T') = \pm \tfrac{1}{4}T,$$

where T is an investigated trial period, n is the number of periods T used in the investigation, and T' is the nearest period to T which should be investigated.[18]

From a study of the distribution of S, Schuster obtains a criterion for judging how large a value of S should be to indicate a true periodicity. As the probability against the reality of a period, he takes the probability that the S value for that period is produced by a random distribution. He finds that the probability of obtaining from a chance distribution a value of S greater than a given one is

$$P_s = e^{-s/\bar{s}},$$

where S is the given value and $\bar{S}$ is the expectancy, or mean value, of S. "The expectancy not depending on the period, we may select for that purpose any portion of the curve in which we have no reason to suspect periodicities."[19]

In my limited application of Schuster's expectancy, I have not found it satisfactory. Its most important defect is that it requires the selection of the probable periodicities before its computation. Let the reader glance for a moment at the periodogram[20] for Dun's ratios which is presented in Figure 2. He will find four conspicuous maxima and two or three minor ones. In advance, we have no good reason for ignoring any of the maxima. If we should, however, consider all of them as indications of probable periods, we cannot find any portion of the curve that is both free from periodicities and long enough for our purpose. In our dilemma we may try to ignore the minor maxima, and take as our expectancy the mean of the S values for periods between 3.05 and 5.82 years. It is apparent that the expectancy will be far below the average level of the curve. Compared with it, the S values at the four conspicuous maxima will all seem very significant and not likely to be produced by a random distribution. By its use we will find that there are only 16 chances in a billion, or one chance in sixty-two and a half million, that the 9.14-year period is not real. As the series of Dun's

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[17] *Ibid.*, p. 193.

[18] *Ibid.*, pp. 192–193; A. Schuster, "On the Periodicities of Sunspots," *Philosophical Transactions of the Royal Society of London*, Series A, CCVI (November, 1906), 71.

[19] A. Schuster, "On the Periodicities of Sunspots," *ibid.*, p. 78.

[20] How the periods that were included in the periodogram were chosen will be considered later.

TABLE II
The Fourier Coefficients and R^2 and S for Each of the
Specified Periods in Dun's Ratios of Table I

i	Period in Years	A	B	R^2	pn	S
21	3.05	−.0486	.0151	.002590	64	.1658
20	3.20	−.0109	−.0113	.000247	64	.0158
19	3.37	.0075	−.0328	.001132	64	.0724
18	3.56	.0213	.0176	.000764	64	.0489
17	3.76	−.0270	−.0291	.001576	64	.1009
16	4.00	−.0191	.0097	.000459	64	.0294
15	4.27	−.0211	−.0084	.000516	64	.0330
14	4.57	−.0178	−.0094	.000405	64	.0259
13	4.92	.0526	−.0090	.002848	64	.1823
12	5.33	.0394	−.0352	.002791	64	.1786
11	5.82	.0108	−.0251	.000747	64	.0478
10	6.40	−.0899	.0378	.009511	64	.6087
9	7.11	−.0213	−.0358	.001736	64	.1111
8	8.00	.0837	−.0292	.007859	64	.5030
7	9.14	.1444	−.0457	.022939	64	1.4681
	10.00	−.0270	−.1504	.023349	60	1.4009
6	10.67	−.0316	−.0506	.003559	64	.2278
	12.00	−.0414	−.1033	.012385	60	.7431
5	12.80	−.0774	−.0480	.008295	64	.5309
	14.00	.0020	−.0086	.000078	56	.0044
	15.00	.0131	−.0762	.005978	60	.3587
4	16.00	−.0519	−.1133	.015531	64	.9940
	17.00	−.1120	−.0504	.015088	51	.7695
	18.00	−.1031	.0307	.011572	54	.6249
	19.00	−.0796	.0679	.010946	57	.6239
	20.00	−.0276	.0624	.004656	60	.2794
3	21.33	.0168	.0195	.000662	64	.0424
2	32.00	.0292	−.0308	.001802	64	.1153
1	64.00	−.0048	.0803	.006471	64	.4141

ratios is very irregular, we may be surprised to find in it such a close
approach to a perfect periodicity as is indicated by this probability.
Considered in relation to other evidence in regard to the reality and
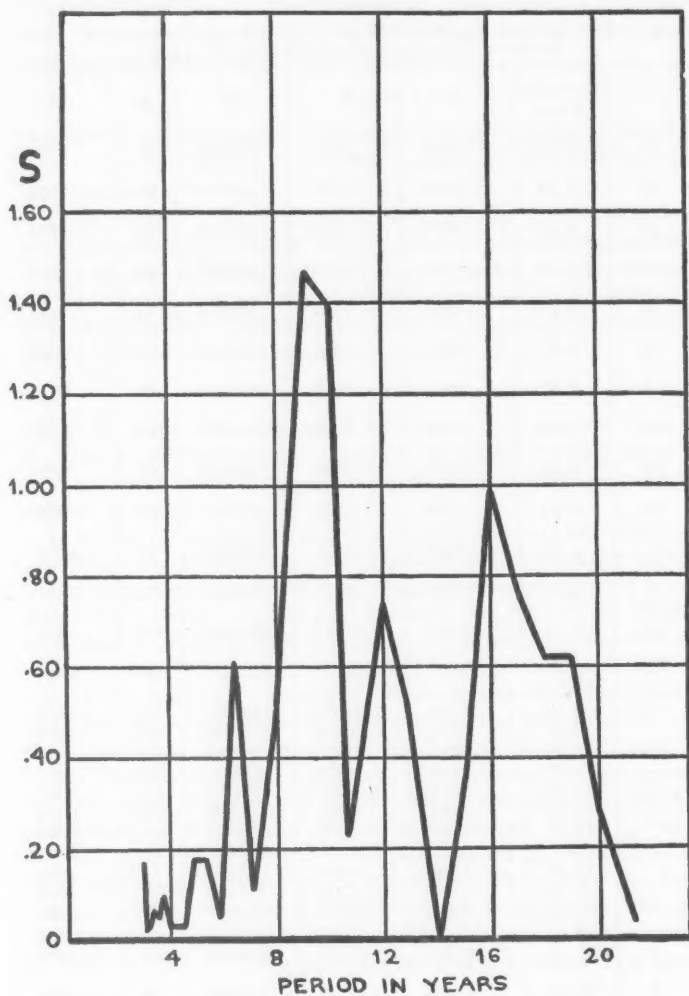
FIGURE 2.—The periodogram for the percentage ratios of business failures to the total number of business concerns in the United States.

continuity of the period, which will be given later, this probability is, however, merely grotesque.

The mean of all the S values in the periodogram would likewise be faulty as an expectancy, for we may disregard in our investigation periods which we consider improbable. Thus, periods below three years are excluded from the periodogram in Figure 2, because, as I shall explain later, they are not present in the given observations. As the S values of the excluded periods are probably very low, the mean S for the whole periodogram is likely to be considerably above the mean for all possible periods.

Let a Fourier series,

$$x = A_0 + A_1 \cos \frac{2\pi r}{N} + B_1 \sin \frac{2\pi r}{N} + A_2 \cos \frac{4\pi r}{N} + B_2 \sin \frac{4\pi r}{N}$$
$$+ A_3 \cos \frac{6\pi r}{N} + B_3 \sin \frac{6\pi r}{N} + \cdots,$$

where x and r are variables and where the A's and B's are parameters, be fitted to N observations, and let N parameters be used in its equation. Its successive harmonics,

$$A_i \cos \frac{2\pi r i}{N} + B_i \sin \frac{2\pi r i}{N},$$

(where $i = 1, 2, 3, \ldots, m$, and where m equals $(N-1)/2$ for odd values of N, or $(N-2)/2$ for even values of N) will have periods of

$$N, \frac{N}{2}, \frac{N}{3}, \ldots, \frac{N}{m}$$

intervals between successive observations. This sequence of periods may be called the complete Fourier sequence. When N is large, as will usually be the case, N/m is approximately 2. While N is the largest period in the given observations, N/m approximates the smallest possible period in them. The series of N observations may contain one instance of the first period in the sequence, two instances of the second, three of the third, etc. The final difference of phase between any of the periods and the one that precedes or follows it in the sequence is thus a full period. As the complete Fourier sequence appears to be representative of all possible periods, I have taken its mean S as the expectancy.

When N is even, the sum of the squares of the deviations of the observed values, x_r, from the Fourier series that gives a perfect fit to the observations is[21]

$$\sum(v^2) = \sum(x^2) - NA_0{}^2 - \frac{N}{2}\sum_{i=1}^{m}(R_i{}^2) - NA^2{}_{m+1} = 0, \text{ where}$$

$$NA_0 = \sum x_r,$$

$$NA_{m+1} = \sum(-1)^r x_r, \text{ and } m = \frac{N-2}{2}.$$

Thus

$$\sum_{i=1}^{m}(R_i{}^2) = 2\left[\frac{\sum(x^2)}{N} - A_0{}^2 - A^2{}_{m+1}\right].$$

Since

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m}S_i = \frac{2}{N-2}\sum_{i=1}^{m}R_i{}^2 N = \frac{2N}{N-2}\sum_{i=1}^{m}R_i{}^2,$$

we have

$$\bar{S} = \frac{4N}{N-2}\left[\frac{\sum(x^2)}{N} - A_0{}^2 - A^2{}_{m+1}\right].$$

The value of $h = S/\bar{S}$ necessary for significance is discussed by Schuster in the following extract, where S is called the intensity of the periodogram:

The probability of an intensity greater than h times the average value is e^{-h}, and we may perhaps begin to suspect a real periodicity when this value is 1 in 200. This gives 5.3 as the value of h . . . which invites further discussion. When h has the value 8, the probability of an intensity greater than h times the expectancy is 1 in 3,000 and we may begin to be more confident that there is some definite cause at work to bring up the periodogram to that value. . . . When h is 16, the chance of being misled by accident is only one in a million.[22]

Another criterion for the reality of a period is given by R. A. Fisher. Let the ratio of the square of the amplitude of a given period to the sum of the squares of the amplitudes for the complete Fourier sequence be

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[21] Brunt, *Combination of Observations*, §§82, 83. In the text I give $\sum(v^2)$ for the special case, $N = 2m+2$. For the general case, $N \geq 2m+1$, where $2m+1$ is the number of parameters,

$$\sum(v^2) = \sum(x^2) - NA_0^2 - \frac{N}{2}\sum_{i=1}^{m}(R_i{}^2),$$

no matter whether N is odd or even.

[22] A. Schuster, "On the Periodicities of Sunspots," *op. cit.*, p. 79.

$$g = \frac{R^2}{\sum_{i=1}^{m} R_i^2} \cdot$$

Then the probability that the largest of the values of g for the Fourier sequence should exceed the given g is

$$P_g = m(1 - g)^{m-1} - \cdots + (- 1)^{m-1}\frac{m!}{k!(m - k)!}(1 - kg)^{m-1},$$

where k is the largest integer less than $1/g$.[23] In order that g be considered significant, P_g must not exceed .05. When P_g is not greater than .05, the first term of the series which equals it gives a good approximation to its value.[24] For practical purposes, we may, therefore, compute the probability by the formula,

$$P_g = m(1 - g)^{m-1}.$$

When Professor Turner examined Schuster's method for investigating periodicities, he came to the conclusion that it involved an unnecessary amount of work. While Schuster proposed that the final difference of phase between successive trial periods should be a quarter-period, Turner suggested that this difference be a full period. He regarded as sufficient the investigation of the Fourier sequence of periods which are exact submultiples of N. His process thus involved the computation of one-fourth of the number of terms required by Schuster's criterion.

Turner argues that the Fourier sequence is both necessary and sufficient. It is necessary, because each term of the Fourier sequence is independent of every other and cannot, therefore, be inferred from any other. It is sufficient, because it will inform us to what extent any periodicity intermediate between two of the Fourier sequence is represented in the observations. Turner proves that if there is an isolated real period between two terms of the Fourier sequence, the two Fourier coefficients, A and B, will both change sign as we pass from the first term to the second. He considers the change of sign important as an indication of a true periodicity.[25] Of the periodicities between two

[23] R. A. Fisher, "Tests of Significance in Harmonic Analysis," *Proceedings of the Royal Society of London*, Series A, cxxv (November, 1929), 58.

[24] *Ibid.*, p. 59. In the table of the values of g for $P = .05$, given there by Fisher, the value of g obtained by the use of the first term alone has in every case its first three digits identical with the corresponding digits of the value of g computed by the exact formula.

[25] H. H. Turner, "On the Expression of Sun-spot Periodicity as a Fourier Sequence," *Monthly Notices of the Royal Astronomical Society*, LXXIII (Supplement, 1913), 715–717.

terms of the Fourier sequence when only one of the Fourier coefficients
or neither of them changes sign, he writes: "wherever we find *changes
of sign* in the Fourier sequence, either of sine or cosine, we may expect
to be able to combine the terms into a simple and more or less perma-
nent periodicity; but *continuities of sign* can only mean spurious perio-
dicities, dual or multiple periodicities, or periodicities which die out and
reappear."[26] In either of these two cases, however, it is best to examine
the observations for possible discontinuities of the period.[27]

For the computation of the terms of the Fourier sequence, it is cus-
tomary to use a number of observations which is an integral multiple
of 4, 8, or 12. Short methods for the computation of the parameters
are then available.[28] In the present study I have omitted an observa-
tion at each end of the series of Dun's ratios. I have thus confined
the investigation of periodicities to the 64 observations for 1868–1931.
The two omitted observations will be inserted again only for fitting
the final curve to the given data.

A survey of Dun's ratios in Figure 1 shows that the time interval
between successive peaks or successive depressions is rarely less than
three years. Yet, if a period of less than three years were present, it
could be represented in the selected 64 observations twenty-one or
more times. Such periods need not, therefore, be considered. I do not
mean to imply that they do not exist but that, if they do, data for
shorter time intervals than the year are required for their study. An-
other limitation of the data gives us the largest period that need be
investigated. Opinions may differ as to how many repetitions of a cycle
are sufficient proof of its existence; but I believe that not many would
generally have faith in a cycle which is contained only twice in the
whole range of observations. The largest period of which we can have
three complete instances in the 64 observations is one of 21.33 years.
Periods exceeding it in length have not been considered.[29]

The method used in this study is a combination of the methods of
Schuster and Turner. The first twenty-one harmonics of the Fourier
sequence have been computed. As the investigation is limited to periods
between 21.33 and 3 years, the harmonics which have periods below
three years were naturally disregarded. However, all periods within
the limits of the investigation which are an integral number of years
in length and differ by over half a year from the nearest period of

[26] Turner, "Further Remarks on the Expression of Sun-spot Periodicity as a
Fourier Sequence," *ibid.*, LXXIV (November, 1913), 18.

[27] Brunt, *Combination of Observations*, pp. 207–208.

[28] *Ibid.*, §§85, 86.

[29] The trend of the given series may be ignored, as it does not depart markedly
from the horizontal.

the Fourier sequence were separately investigated.[30] The results for all the studied periods are in Table II; the periodogram showing the value of S for each of them is in Figure 2.

The mean S of the complete Fourier sequence for the 64 observation is

$$\overline{S} = .1961.$$

To satisfy Schuster's minimum requirement for significance,

$$P_s = e^{-s/\overline{s}} = .005,$$

we need a value of S at least as great as

$$S = 5.3\overline{S} = 1.0393.$$

From Fisher's requirement that

$$P_g = m(1 - g)^{m-1}$$

should not exceed .05, it follows when

$$m = \frac{N - 2}{2} = \frac{64 - 2}{2} = 31$$

that

$$g = \frac{R^2}{\sum_{i=1}^{m} R_i{}^2} = .1929,$$

or a value exceeding it, is necessary for significance. Substituting

$$\sum_{i=1}^{m} R_i{}^2 = .0950$$

in the last equation, we obtain

$$R^2 = .1929(.0950) = .018326.$$

In the computation of the $R_i{}^2$ of the Fourier sequence, 64 observations have always been used. Hence,

$$S = R^2 N = (.018326)64 = 1.1729$$

is for a period of this sequence the minimum that will satisfy Fisher's test.

The only S values which meet either of the two tests for significance are those for the 9.14- and 10.00-year trial periods. In the periodogram, we observe that the highest of the maxima is the one for the 9.14-year period. This period has, therefore, been selected for study. Its S value

[30] By the method which Brunt gives, *ibid.*, §87.

is considerably above the required minimum of either test. In Table II, we note that A changes sign as we pass from the sixth to the seventh harmonic but that B does not. As there is only one change of sign in favor of the selected period, a study of its possible discontinuities is indicated.

IV. A CLOSER ANALYSIS OF THE 9–10 YEAR CYCLE

In the investigation of discontinuities, all possible ten-year[31] ranges within the 64 observations are analyzed harmonically. For computing

the A's we require the moving sum, $\displaystyle\sum_{r=u}^{u+9} x_r \cos \frac{2\pi r}{10}$, and for computing the

B's we need $\displaystyle\sum_{r=u}^{u+9} x_r \sin \frac{2\pi r}{10}$, where u takes the values 0, 1, 2, $\cdots$, 54.

Each sum can be computed independently of all the others. The arithmetical work may, however, be greatly reduced, if it is done by the usual moving average technique.

Each of the above sums, when multiplied by 2/10, yields a Fourier coefficient for one of the ten-year ranges. The phase of maximum for each period is computed by the formula,

$$\phi = \text{arc tan } \frac{B}{A},$$

where A and B are the Fourier coefficients for the period, and ϕ is its required phase.[32] The phase for each ten-year period is shown in Figure

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[31] It is for the present immaterial whether the true period is nearer nine years or nearer ten. Turner, in his study of discontinuities, writes: "Take a single *arbitrary* period, in this case 12 years. Sunspots are suspected of periods *near* this, whether 8 years or 14 years, etc., does not matter for our present purpose." H. H. Turner, "On a Simple Method of Detecting Discontinuities in a Series of Recorded Observations," *Monthly Notices of the Royal Astronomical Society*, LXXIV (Dec., 1913), 83.

[32] Brunt, *Combination of Observations*, p. 198. The equation,

$$x = A_0 + A \cos \phi + B \sin \phi,$$

where ϕ is given by the formula above, denotes both the maximum and minimum ordinates of the Fourier curve. When x is the maximum ordinate, however,

$$A \cos \phi + B \sin \phi > 0.$$

From this condition and the formula for ϕ, it follows that the phase of maximum is an angle in the

$\left\{\begin{array}{l}\text{first quadrant, when } A \text{ and } B \text{ are both positive, or} \\ \text{third quadrant, when } A \text{ and } B \text{ are both negative;} \\ \text{second quadrant, when } A \text{ is negative and } B \text{ is positive, or} \\ \text{fourth quadrant, when } A \text{ is positive and } B \text{ is negative.}\end{array}\right.$

It should be noted that the addition of 2π, or multiplies of it, to the principal

3. The ordinate of the first point to the left is the phase for the period 1868–1877; the ordinate of the second point is the phase for the period 1869–1878; and so on.

If a period of ten years were present throughout the observations, the phase would be constant. If a different period were continuously present, the change of phase from each ten-year period to the next would be constant, that is, the computed phases would distribute themselves along a straight line. If one period were present throughout part of the observations and another throughout a second part, the phases for the first part would distribute themselves along one straight line and those for the second along another. In any case, the true period during the time interval to which each line applies may be deduced from the formula[33]

$$T' = \frac{T}{1 - \dfrac{T\beta}{2\pi}}$$

Here β is the average change of phase in degrees, or the slope of the fitted line, T is the primary period[34] analyzed, and T' is the true period during the given time interval.

In Figure 3, however, we find no linear distributions of successive phases that are long enough to include the change of phase during a single complete instance of the selected period. The longest continuous interval to which a straight line may be fitted is the one for 1879–1884; the slope of the fitted line yields a period of 9.62 years. As the time interval to which the β applies is not even two-thirds the length of a single instance of the period which is obtained by means of it, the deduced period is of doubtful significance. Other continuous time intervals to which straight lines may be fitted are even shorter than the one given. I hesitate, therefore, to state the periods found for them.

The scatter of the phases in Figure 3 shows that the selected period is highly discontinuous. It is, indeed, a question whether it does not deny the presence of the periodicity altogether. Let us consider, however, the probabilities against the reality of the 9.14-year period,

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values of some of the phases may reveal a continuity of phase through time, which would otherwise not be apparent. A diagram of the type illustrated in Figures 3 and 4 would generally be repeated vertically for several periods of 2π. As the repetition would, in the present case, not affect materially the conclusions, it was left out.

[33] *Ibid.*, pp. 209–210. The solution of the two examples there may be expressed by this formula.

[34] The ten-year period is, in the present case, the primary period.
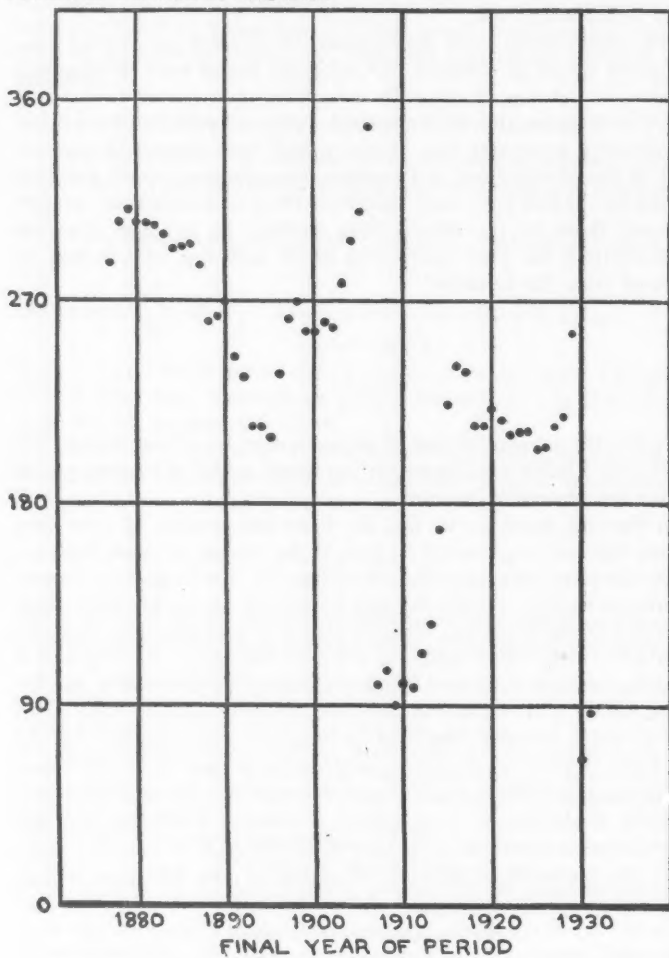
FIGURE 3.—The phase of maximum for each period of ten years
having the given final year.

$$P_s = .000559 = \frac{1}{1,790},$$

and

$$P_g = .00777 = \frac{1}{129}.$$

There is only one chance out of 1,790 that the S value of the period is produced by a random distribution of all the S values; there is only one chance out of 129 that its g value should be exceeded by the largest g of the Fourier sequence. That the high S and g values are entirely the result of chance is possible but not probable.

Conflicting evidence like the present tests the fairness of the investigator. One who does not believe in the presence of the periodicity will stress the evidence of the study of discontinuities. Relying on Figure 3, he may attribute the high values of S and g to chance. On the other hand, one who has faith in the periodicity will stress the low probabilities against its reality. He may explain that the irregularities of the original data obscure the presence of the periodicity and prevent its appearance in Figure 3.

It is pertinent to inquire whether an hypothesis which gives due weight to the two conflicting sources of evidence can be constructed. Let us contrast the irregularity of the given data with the characteristic regularity of the graph of a single harmonic. Obviously, the data do not conform to one of the properties of the selected periodicity; they may, however, conform to another. In the study of discontinuities, individual observations have great weight; one irregular observation among the ten used for the computation of each phase may influence the result considerably; and the irregularities are many. The nonconformity of the data to at least one property of a simple periodicity is disclosed by the study of discontinuities. On the other hand, in the computation of S and g, where all the 64 observations are used, the irregular elements may compensate each other to some extent; conformity to some other periodic property may then manifest itself.[35]

Our best approximation to the length of the selected period is at present 9.14 years. A closer approximation to its actual length may be found by secondary analysis. Methods for that purpose are offered by both Turner and Schuster. The former, however, derives his formula on the assumption that the actual period is isolated between two successive terms of the Fourier sequence. As there is only one change of sign from the sixth to the seventh harmonic and as no isolated period is revealed in the study of discontinuities, we have at present no justification for such an assumption.

[35] Evidence in favor of the hypothesis will be considered later.

The process of secondary analysis by Schuster's method is similar to the one by which discontinuities have been studied. Instead, however, of finding the phase for successive periods, each of which differs from the preceding in only a single observation, we get the phase for periods which are distinct from each other in all their observations. In the present case, we divide the first 60 of our 64 observations into six periods. The first period consists of the first ten observations; the second includes all the observations from the eleventh to the twentieth; the third includes all the observations from the twenty-first to the thirtieth; and so on. Each period is analyzed harmonically, and its phase of maximum is found by the formula,

$$\phi = \text{arc} \tan \frac{B}{A}.$$

Usually a straight line is fitted to the phases; its slope, β, is the average change of phase from one period to the next. The period obtained by secondary analysis is[36]

$$T' = \frac{T}{1 - \dfrac{\beta}{2\pi}},$$

where T is the primary period analyzed.

The six computed phases are presented in Figure 4. Obviously, no straight line will give a good fit to all the six points. We might, perhaps, ignore the erratic fourth point and fit by inspection a straight line to the others. Its slope would approximate $-15°$ and yield a period of 9.6 years. The neglect of the phase for the time interval, 1898–1907, would introduce, however, an unknown amount of error. It would lead to a period which is based on only 50 observations and unaffected by the intervening ten observations. For accuracy and reliability, it is, therefore, best to employ a more thorough method of secondary analysis.

Since there is a change of sign between the sixth and seventh harmonic, the periodicity should, by Turner's criterion, be located between the limits of these two of the Fourier sequence. At present we do not know whether the periodicity is single or multiple. By a given criterion, two periods begin to be independent when the final difference of phase between between them is a quarter-period. If we divide the 64 observations by 7, $6\frac{3}{4}$, $6\frac{1}{2}$, $6\frac{1}{4}$, and 6, we obtain all the periods

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[36] Brunt, *Combination of Observations*, p. 197. In Brunt's formula, 2π is multiplied by m. As the m of the formula is unity in the present case, it is not introduced.

PHASE OF PERIOD IN DEGREES
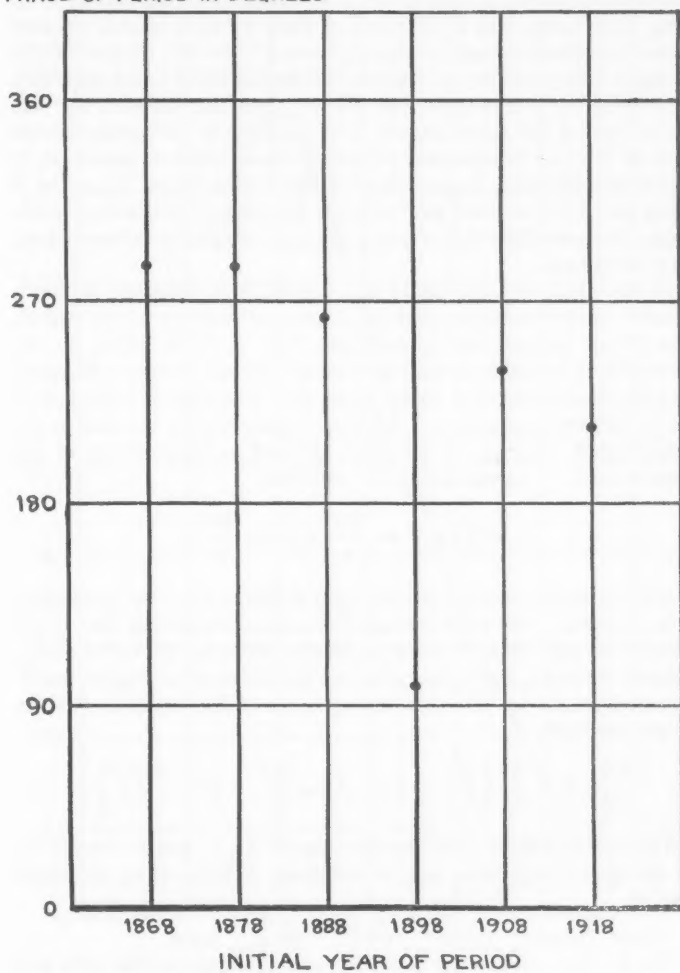
FIGURE 4.—The phase of maximum for each period of ten years
having the given initial year.

within the limits of the sixth and seventh harmonic which are to some extent independent of each other. Their lengths are respectively 9.14, 9.48, 9.85, 10.24, and 10.67 years. If their S values should arrange themselves about a single maximum, we may know that the periodicity is single. The possibility of a second maximum which is not indicated by any of the five periods need not be taken into account. For the period having this maximum would not begin to be independent from each of the two investigated periods between which it would lie; it could not, therefore, be considered distinct from them. When the S value for neither of these two indicates the presence of a second maximum, the possibility that a maximum is intermediate between them may be ignored.

In the harmonic analysis of any period, it is customay to use a number of observations which is an integral multiple of the period. The largest integral multiples of 9.48, 9.85, or 10.24, which do not exceed 64, the number of our observations, are not, however, integers. As a fractional number of observations cannot be used, the usual practise must here be abandoned. All the 64 observations are used in the computation, although 64 is not an integral multiple of any of the three periods. A curve having the equation,

$$x_r = A_0 + A \cos \frac{2\pi r}{p} + B \sin \frac{2\pi r}{p},$$

where p is the length of the investigated period, is fitted by the method of least squares. An examination of the equation reveals that it expresses the regression of the dependent variable, x_r, on the two independent variables, $\cos 2\pi r/p$ and $\sin 2\pi r/p$. Either by analogy to multiple correlation or by the actual minimization of the sum of the squares of the residuals,

$$\sum_{r=0}^{N-1}(v_r{}^2) = \sum_{r=0}^{N-1}\left[\left(x_r - A_0 - A \cos \frac{2\pi r}{p} - B \sin \frac{2\pi r}{p}\right)^2\right],$$

and its partial differentiation with respect to A_0, A, and B, respectively, the normal equations may be obtained. S is found by the usual formula,

$$S = R^2 N = (A^2 + B^2)N.$$

The S values for the five periods within the limits of the sixth and seventh harmonic arrange themselves about a single maximum. I have investigated several periods near this maximum in order to improve my estimate of the length of the period that has the highest S value. Table III and Figure 5 present the results of the secondary analysis. Both show that the 9.4-year period has the highest S value.
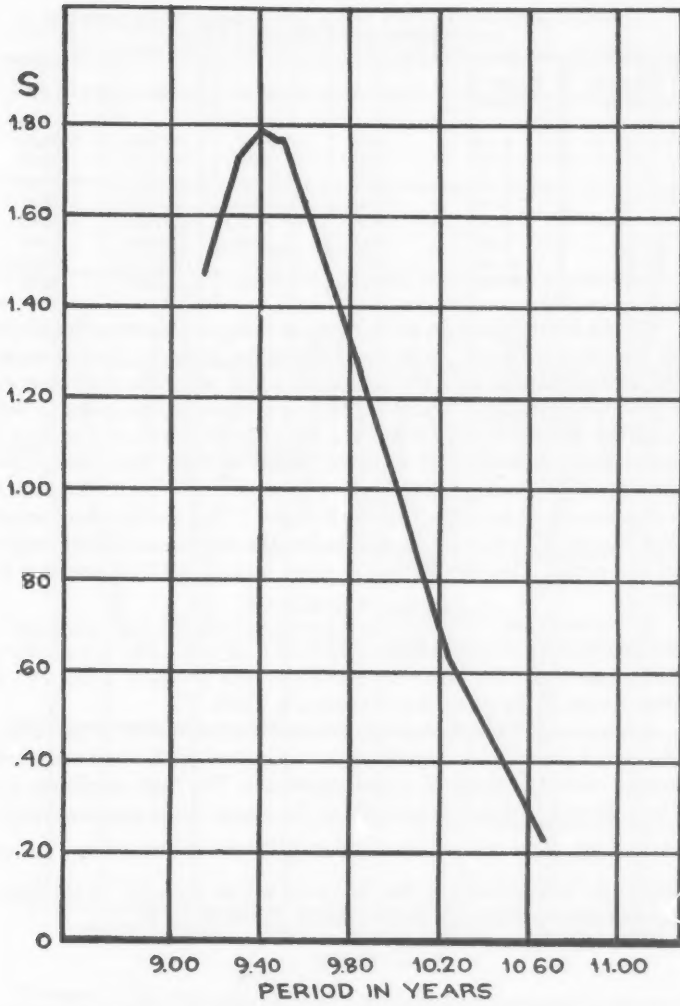
FIGURE 5.—The values of S for periods ranging in length from 9.14 to 10.67 years.

TABLE III
THE FOURIER COEFFICIENTS AND R^2 AND S FOR PERIODS RANGING
IN LENGTH FROM 9.14 TO 10.67 YEARS

Divisor of 64	Period in Years	A	B	R^2	S
7	9.14	.1444	−.0457	.022939	1.4681
	9.30	.1314	−.0990	.027067	1.7323
	9.40	.1121	−.1241	.027967	1.7899
$6\frac{3}{4}$	9.48	.0911	−.1391	.027648	1.7695
	9.50	.0874	−.1414	.027633	1.7685
$6\frac{1}{2}$	9.85	.0019	−.1402	.019660	1.2582
$6\frac{1}{4}$	10.24	−.0373	−.0921	.009873	.6319
6	10.67	−.0316	−.0506	.003559	.2278

The 64 observations do not contain an integral number of instances of the 9.4-year period. As 66 observations do, however, contain seven complete instances of a 9.43-year period, I now reintroduce the omitted ratios of failures for 1867 and 1932. Each of these two ratios fits in well with the ratios near it. Considered together, the two are less than a quarter of a period. It is, therefore, highly unlikely that their introduction should change the length of the period with the highest S value to such an extent as to make it distinct from the 9.4-year period. For the 66 observations, as well as for the 64, the estimated length of the period with the highest S value may be safely taken as 9.43 years.

In Figure 6 are presented the ratios of failures for 1867–1932, with a fitted periodic curve, having a period of 9.43 years. The actual ratios and those computed from the curve are both given in Table IV. A description of the curve may be found in Table V.

Section (a) of Table V shows that two of the parameters of the curve, A_0 and A, are highly significant as compared with their standard errors, while the third, B, is not significant. The high amplitude for the 9.43-year period is due chiefly to the size of A. As may be seen in section (c), R^2 is almost one-third of $\sum R_i^2$ for the complete Fourier sequence. The probability that this ratio will be exceeded by the highest g of the Fourier sequence for the 66 observations is

$$P_g = .000181 = \frac{1}{5520}.$$

The S value for the period is ten and one-third times the expectancy; the probability that it will be exceeded in a random distribution of all the S values is

$$P_s = .0000326 = \frac{1}{30,700}.$$

PERCENTAGE RATIO OF FAILURES

ACTUAL DATA

$X_r = .9985 + .1773 \cos \frac{14\pi r}{66}$

$-.0442 \sin \frac{14\pi r}{66}$

YEAR
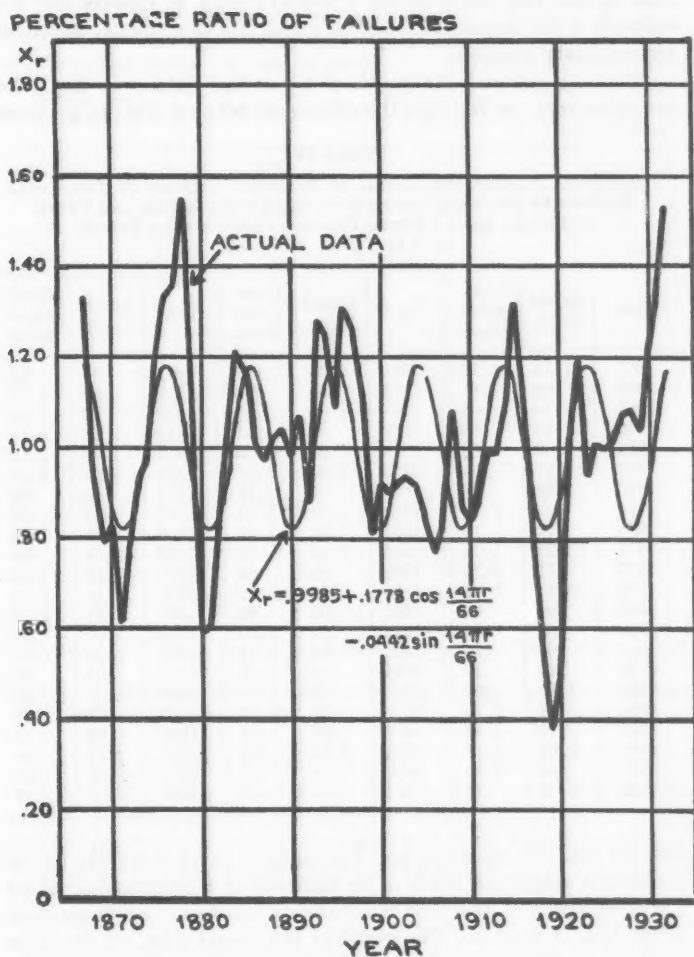
FIGURE 6.—The percentage ratios of business failures to the total number of business concerns in the United States, 1867–1932, with a fitted periodic curve.

When these very low probabilities are compared with the much higher ones against the reality of the 9.14-year period, it appears that our estimate of the length of the actual period has been greatly improved by secondary analysis.

When we compare the graph of the actual data with the fitted periodic curve, we find that the differences between the two are more

TABLE IV

THE ACTUAL PERCENTAGE RATIOS OF BUSINESS FAILURES TO THE TOTAL NUMBER OF BUSINESS CONCERNS IN THE UNITED STATES AND THOSE COMPUTED FROM A FITTED PERIODIC CURVE WITH A PERIOD OF 9.43 YEARS, 1867–1932

Year	Actual Ratio	Computed Ratio	Year	Actual Ratio	Computed Ratio	Year	Actual Ratio	Computed Ratio
1867	1.33	1.18	1889	1.04	.87	1911	.88	.95
1868	.94	1.11	1890	.98	.82	1912	.99	1.07
1869	.79	1.00	1891	1.07	.84	1913	.99	1.16
1870	.83	.88	1892	.88	.93	1914	1.10	1.18
1871	.61	.82	1893	1.28	1.05	1915	1.32	1.12
1872	.77	.83	1894	1.25	1.15	1916	1.00	1.01
1873	.93	.92	1895	1.09	1.18	1917	.80	.90
1874	.97	1.03	1896	1.31	1.14	1918	.58	.83
1875	1.20	1.14	1897	1.26	1.03	1919	.38	.83
1876	1.33	1.18	1898	1.10	.91	1920	.49	.90
1877	1.36	1.15	1899	.81	.83	1921	1.02	1.02
1878	1.55	1.05	1900	.92	.82	1922	1.19	1.13
1879	.95	.93	1901	.90	.89	1923	.94	1.18
1880	.59	.84	1902	.93	1.00	1924	1.01	1.16
1881	.61	.82	1903	.94	1.11	1925	1.00	1.07
1882	.83	.87	1904	.92	1.18	1926	1.01	.95
1883	1.06	.98	1905	.85	1.16	1927	1.07	.85
1884	1.21	1.10	1906	.77	1.08	1928	1.08	.82
1885	1.16	1.17	1907	.83	.96	1929	1.04	.86
1886	1.01	1.17	1908	1.08	.86	1930	1.21	.96
1887	.97	1.10	1909	.87	.82	1931	1.33	1.08
1888	1.02	.98	1910	.84	.85	1932	1.53	1.17

striking than the resemblances. The periodic curve is regular. It has a constant amplitude. Each of its half-periods is symmetrical about a central ordinate. The time interval between its successive maxima, or minima, is constant. The graph of the actual data, on the other hand, is very irregular. It has a highly variable amplitude for its various cycles. It exhibits no marked symmetry of any kind. The timing of its peaks and depressions does, however, generally correspond well to the timing of the maxima and minima of the periodic curve. Among the four specified properties of the mathematical periodicity, we find only one to which the actual data do generally conform.

<div align="center">

TABLE V

DESCRIPTION OF THE FITTED PERIODIC CURVE

(a) The Parameters and Their Standard Errors

</div>

Symbol for Parameter	Parameter	Its Standard Error	Numerical Ratio of Parameter to Standard Error
A_0	.9985	.0236	42.31
A	.1778	.0334	5.32
B	− .0442	.0334	1.32

<div align="center">

(b) Standard Error of Curve

$\epsilon = .192$

(c) Measures Pertaining to Period of Curve

Period of Curve = 9.43 Years

</div>

$$R^2 = .033567 \qquad\qquad S = 2.2154$$

$$\sum_{i=1}^{32} R_i^2 = .1040^* \qquad\qquad \bar{S} = .2145^*$$

$$g = \frac{R^2}{\sum_{i=1}^{32} R_i^2} = .3228 \qquad\qquad \frac{S}{\bar{S}} = 10.33$$

$$P_g = .000181 \qquad\qquad P_s = .0000326$$

<div align="center">

(By Fisher's Test)   (By Schuster's Criterion)

</div>

* Both $\sum_{i=1}^{32} R_i^2$ and $\bar{S}$ are for the Fourier sequence of the 66 observations; the former is the sum of all the R_i^2 for that sequence, and the latter is the mean of all the S_i for the same sequence.

The given comparison does not confirm the presence of the mathematical periodicity in the data. It does, however, bear out the conclusion that 9.4 years may be regarded as the typical duration of the cycle in the ratios of failures for 1867–1932, if by typical duration be understood the length which is most representative of the variable actual lengths of the cycles. My conclusion may be examined in relation to estimates of the average duration of the business cycle. Tugan-Baranovski and Bouniatian give this average duration as ten years, while Moore, Pigou, and Lavington give it as eight years.[37] My estimate for the cycle in business failures is between these two, but is closer to the first. Since "Tugan-Baranovski takes 7 to 11 years as the limits of variation in the length of cycles," and "Bouniatian says that 'under

[37] Wesley C. Mitchell, *Business Cycles; the Problem and Its Setting* (1927), pp. 384–385.

normal conditions' cycles last from 9 to 11 years,"[38] my estimate is fully within their specified limits. Moore writes: "Because a natural eight-year cycle has been isolated is no reason for denying the possible existence of other natural cycles, major or minor."[39] Consequently, there is no conflict between his conclusion and mine. If Pigou and Lavington admit the possibility of variation in the duration of business cycles, as they probably do, my estimate is consistent with theirs. If they do not, my conclusion may, nevertheless, not conflict with theirs, as the cycle in business failures and the business cycle are, while closely related, not necessarily of identical length.[40]

Finally, I shall consider the uses that may be made of my estimate. The economic historian may find the typical duration of cycles in business failures during the past 66 years of historical interest. The business cycle theorist may find the relation of my estimate to estimates of the average duration of business cycles worthy of note. For the practical economist, however, who seeks means for forecasting, I have little to offer. When I accept 9.4 years as the typical duration of the cycle in business failures, I am not assuming that the period is constant. If I should do so, Figure 6 would not support my assumption. The graph of the actual data has, for instance, peaks for 1893 and 1896, while the periodic curve combines the two peaks into a single maximum which actually coincides in its timing with the depression for 1895. As the period may be variable, even its typical duration in the future need not be 9.4 years. Moreover, seven instances of a cycle are hardly sufficient as a basis for forecasting. If, during the next 66 years, we find that the typical duration remains unchanged and that certain known causes keep it constant, we shall have grounds for prediction. For the present we must wait.

Chicago, Illinois

[38] *Ibid.*

[39] Moore, *Generating Economic Cycles*, p. 11.

[40] Under the influence of articles by W. L. Crum and Joseph Kitchin, a 40-month cycle has been popularized (Mitchell, *op. cit.*, p. 385). Both my periodogram and my Fourier sequence of Table II give little evidence of the existence of such a cycle. It is necessary to remark, however, that Crum used monthly data in his analysis of the rates on commercial paper, where he found evidence of this cycle. I suppose that Kitchin also used in his study observations for time intervals shorter than the year. For the study of cycles of that length, such observations are appropriate.

THE PERIOD OF PRODUCTION UNDER CONTINUOUS INPUT AND POINT OUTPUT IN AN UNPROGRESSIVE COMMUNITY[1]

By C. H. P. Gifford

THIS essay is intended as a study in the structure of production; and it sets out in particular to show how the rate of interest, the length of the period of production (which may be regarded as the production factor Time), and output per head, are connected with one another. The simplifying assumptions which I shall adopt arise naturally from the conditions of the problem.

There are six main assumptions:

1. That the community is unprogressive, that is, that its technical knowledge remains unaltered. This assumption is required because the structure of production is here discussed as a static, not a dynamic, problem.

2. We assume that the population is constant and tastes unchanged.

3. We assume the community to be in equilibrium, that is, that unemployment is unknown. Such profits as will lead entrepreneurs to offer full employment are treated as remuneration to them for their labor and so as an element in cost; and the price of any commodity is assumed equal to its cost.

4. It follows from the above that the wealth of the community must remain unaltered; this corresponds to the factor Land in traditional economic theory. I propose to abstract from it in this paper, since it would seem reasonable to suppose that light can be thrown on the economic organization of a country today even if we neglect the wealth possessed by its inhabitants in 1250 B.C.

5. A *Unit of Labor* is defined as equal to a unit of population (i.e, one man) multiplied by a unit of time. This involves either assuming that all men are equally remunerated for equally valuable work, or else reckoning an entrepreneur who receives n times the remuneration of an unskilled laborer to be equal to n units of population. The reader may choose for himself the construction he prefers. Whichever is chosen, we have now reduced the factors of production to two which are quantitative, Labor and Time.

6. Much labor is embodied in the form of *durable* producers' goods

[1] Though the exposition has been considerably modified, this paper may be regarded as an expansion of my article on "The Concept of the Length of the Period of Production" in the *Economic Journal* of December 1933. I have not hesitated to repeat myself, however, when necessary to achieve continuity in the argument of this paper.

such as machines and factories, each of which during its lifetime helps in the making of a flow of consumers' goods. It would be inconvenient if we had to say that the length of the period of production of a given consumers' good depended on the particular moment at which it passed through a certain machine or factory. So we shall proceed as if the output of the machine or factory and the labor were concentrated at a particular point of time.[2] This we may call the condition of Point Output. But the input will be assumed to be a continuous stream.

We are now in a position to begin our analysis. For convenience I shall write of a particular commodity, the *Chocolate*.

R, the length of the period of production, and L, the total labor cost of a chocolate, may be defined as follows:

Let $A(x)$ be the number of men working per unit of time at time $t-x$ on a chocolate finished at time t. Then we have, by definition,

$$L = \int_0^\infty A(x)dx$$

and

$$R = \frac{\int_0^\infty x \cdot A(x)dx}{\int_0^\infty A(x)dx}.$$

R may thus be described as the "average delay period" between the time when the labor was expended and the time when the finished chocolate emerged from the productive process.

Each function $A(x)$ represents a possible organization of production, and for mathematical convenience I shall assume that in every case this function is continuous. This is the condition of Continuous Input. It may also be well to explain that, if in a unit of time at time X α men are engaged in building a factory which helps to make β chocolates, we regard α/β units of labor as having been expended on each of these chocolates at time X.

We can now find a formula for the cost of production of a chocolate. Let π be the cost to the entrepreneur of a chocolate finished at time t. Let W be the time-rate of wages, and I the rate of interest. Simple interest will be assumed.[3] Then, on account of the $A(x)$ units of labor

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[2] The point of time is chosen as follows: Let the machine or factory begin operations at time T_0. Then, if $N(x)$ be the output produced with its aid at time T_0+x, the point of time will be T_0+T, where

$$T = \frac{\int_0^\infty x N(x)dx}{\int_0^\infty N(x)dx}.$$

The point of time (not necessarily the same) at which the labor applied to it is assumed to be concentrated is determined by a similar construction.

[3] See note 6.

expended on the chocolate in a unit of time at time $t-x$, the entrepreneur will incur a cost of

$$W \cdot A(x)(1 + Ix).$$

We have, therefore,

$$\pi = W \int_0^\infty A(x)dx + WI \int_0^\infty x \cdot A(x)dx$$
$$= WL(1 + RI). \tag{1}$$

Now the individual entrepreneur, who as a rule has to take both W and I as outside his control, will always try to organize the process of production in such a way that his money-costs are minimized. Consider, then, all those A-sets (i.e., A functions) for which $R = R_0$. Since W and I must both be positive, it is clear from equation (1) that that particular A-set among them which yields the smallest value for L will yield also the smallest value for π. We can, therefore, eliminate all other A-sets for which $R = R_0$ as being organizations of production which could not occur in actual practice; and in this way we are left with a class of A-sets which we may call ω, such that to any value of R corresponds one and only one value of L.[4] L can thus be regarded as a function of R, which I shall from now on write as $L(R)$; and $L(R)$ will represent the technical knowledge of the community, which is given.[5] I shall assume for mathematical convenience that $L(R)$ is a continuous function of R.[6]

We must now discover which member of ω will actually be chosen. W and I are given, and the entrepreneur can choose the value of R. He will naturally take that R which minimizes his costs, i.e., which

[4] We could still further reduce the number of possible A-sets by the rule that, if more than one member of ω yields a particular value of L, that member alone can be chosen which yields the smallest value for R; but no useful purpose would be served by so doing.

[5] It would be strictly more accurate to say that ω represents the technical knowledge of the community; but that aspect of ω which is given by $L(R)$ is all that is needed for the purpose of this essay.

[6] It was in order to secure this reduction of the technical conditions to a convenient form that the assumption of simple interest was made. If we assume that interest is continuously compounded, we have

$$\pi = W \int_0^\infty A(x) \cdot e^{Ix}dx,$$

which is not manageable. The assumption of simple rather than compound interest may be expected, however, in the absence of proof to the contrary, to lead to differences of detail and not principle in the results obtained.

gives $\dfrac{\partial \pi}{\partial R} = 0$, and $\dfrac{\partial^2 \pi}{\partial R^2}$ positive for $\dfrac{\partial \pi}{\partial R} = 0$. These conditions give us

$$\frac{\dfrac{d}{dR}\{L(R)\}}{L(R)} = -\frac{I}{1+RI} \tag{2}$$

and

$$L(R) \cdot \frac{d^2\{L(R)\}}{dR^2} > 2\left\{\frac{dL(R)}{dR}\right\}^{2 \;\;[7]}. \tag{3}$$

Let us now write output per head $= V = 1/L(R)$. V is thus a function of R, and the functional relationship between them, which represents the community's technical knowledge, is taken as known.

Let the rate of real wages $= E = W/\pi$, by assumption (3). Our three equations can then be written:

$$V = E(1 + RI) \tag{α}$$

$$\frac{dV}{dR} = EI \quad [8] \tag{β}$$

$$\frac{d^2V}{dR^2} < 0. \quad [9] \tag{γ}$$

From (β) we may deduce that entrepreneurs will not pay a positive rate of interest unless more roundabout methods of production yield an increase in output per head; and (γ) is the condition that more roundabout methods of production shall obey a Law of Diminishing Return.

Since the functional relationship between V and R is known, equations (α) and (β) are sufficient to enable us to find R, V, and E, whenever the value of I is given.

If we make the necessary changes of notation, equations (α) and (β) are the same as equations (13) and (14) set out by Wicksell on pp. 96, 97 of his *Uber Wert, Kapital und Rente*;[10] but he had only defined 'die durchschnittliche Länge der Kapitalinvestierung'—his name for which

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[7] For the details of the proof see Gifford, *loc. cit.*, p. 615.

$$\frac{d\{L(R)\}}{dR} = \frac{d\{L(R)\}}{dV} \cdot \frac{dV}{dR} = -\frac{1}{V^2}\frac{dV}{dR}.$$

$$L(R)\frac{d^2}{dR^2}\{L(R)\} - 2\left\{\frac{dL(R)}{dR}\right\}^2 = -\{L(R)\}^3\frac{d^2}{dR^2}\left\{\frac{1}{L(R)}\right\}.$$

[10] Reprinted from the 1893 edition by the London School of Economics in the "Scarce Tracts in Economic and Political Science" series.

I have called the length of the period of production—for the special case of those A-sets for which $A(x)$ is constant for $0 < x < \alpha$, where $t - \alpha$ is the time at which production is begun, and zero for $\alpha < x < \infty$. In other words, when there is a definite time at which work is started on the commodity, and a constant labor force is then steadily employed on it until it is finished.

Wicksell, however, arrived at his results by different methods; and it is not without interest to compare his procedure with my own. He considered two distinct problems. First, he took the case in which the workmen are also the entrepreneurs, and, the rate of interest being given, aim at minimizing their real wages. Here, though under modern industrial conditions the application may be largely limited to agriculture and retail trading, the argument is valid; and it will be noted that if in my analysis we differentiate equation (α) w.r.t. R, assuming

I constant, we get $\dfrac{dV}{dR} = \dfrac{dE}{dR}(1+RI) + EI$, and combining this with

(β) we have $\dfrac{dE}{dR} = 0$. That is, for any rate of interest, the entre-

preneur, in seeking to minimize his money costs, will choose that length of the period of production which makes real wages a maximum.[11]

Wicksell then proceeds to consider[12] the case in which the entrepreneur supplies his own capital and, the rate of real wages being given, aims at maximizing the rate of interest which he receives upon it; and this leads to the same results. But such a set-up is not permissible; for the rate of real wages depends on the length of the period of production, and cannot be known when the latter has still to be determined.

Wicksell clarifies his results, however, by a very elegant geometrical construction. Substituting my notation for his, I give a free translation of the paragraph in which he sets it out. Reference should be made to Figure 1.

In order that this conception may be more easily grasped, we shall illustrate the result geometrically. Take R as the abscissa and V as the ordinate of a curve drawn to represent the known properties of V. This curve must be concave to-

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[11] It can easily be shown that E is a maximum and not a minimum. For, for I constant and $dE/dR = 0$, we have

$$\frac{d^2V}{dR^2} = \frac{d^2E}{dR^2}(1 + RI) \qquad \text{and} \qquad \frac{d^2V}{dR^2} \quad \text{is negative by}$$

equation (γ).

[12] *Op. cit.*, p. 98.

wards the x-axis,[13] and (since even instantaneous production would yield some output) must show V positive for $R = 0$. If we take a point on this curve and draw the straight line from it to the point $(-1/I, 0)$, it will cut the y-axis at the point $(0, E)$, as will be immediately obvious if we write equation (α) in the form

$$\frac{V}{E} = \left(\frac{1}{I} + R\right)\Big/\frac{1}{I}\cdot$$

The maximum of E will be reached if we draw the tangent from $(-1/I, 0)$ to the curve, as shown by equation (β).[14]
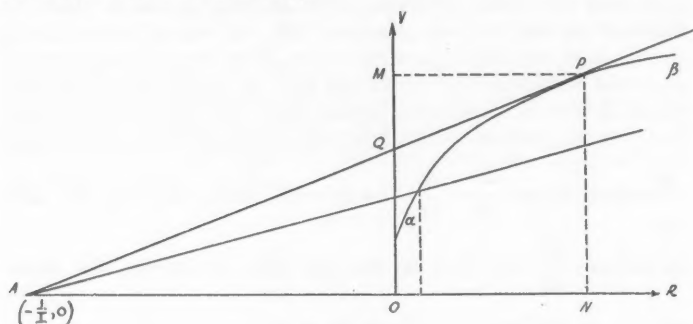
FIGURE 1

It may now be of interest to solve equations (β) and (γ) for a particular form of V, deriving from them the functional relationships between R, V, or E and I.

Consider, first, the general case where

$$V^n = a + bR \qquad n, a, \text{ and } b \text{ being constants}$$

Then,

$$n \log V = \log (a + bR),$$

$$\frac{n}{V}\frac{dV}{dR} = \frac{b}{a + bR}.$$

But, $\quad \dfrac{1}{V}\dfrac{dV}{dR} = \dfrac{I}{1 + RI}$, dividing (γ) by (β).

Hence,

$$R = \frac{1}{(n-1)I} - \frac{an}{b(n-1)},$$

---

[13] Because $d^2V/dR^2 < 0$, by (γ).

[14] *Op. cit.*, p. 97.

and

$$\frac{dR}{dI} = -\frac{1}{(n-1)I^2}.$$

That is, if V is of the form $V^n = a + bR$, the more sensitive V is to changes in R, the more sensitive R will be to changes in I. This should be borne in mind when judging the plausibility of the actual example chosen.
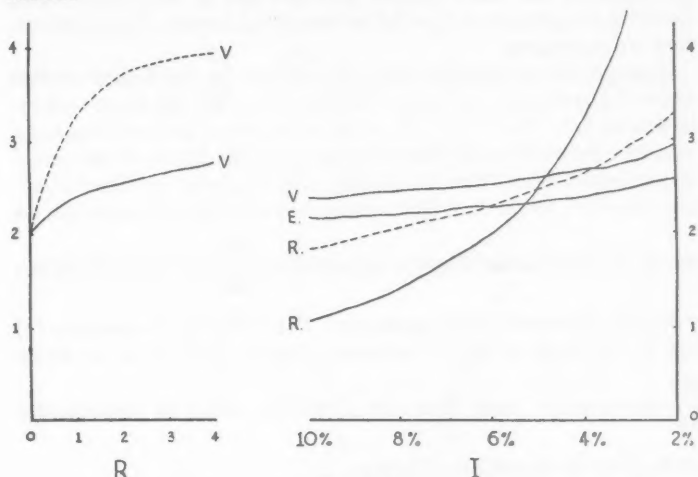
FIGURE 2

Let the unit of time be one year, the unit of output 10^5 chocolates, and let us suppose technical conditions to be represented by the function

$$V^8 = 2^8\left(1 + \frac{16R}{5}\right).$$

Then, working as above, we get

$$7R = \frac{1}{I} - \frac{5}{2},$$

from which we have

$$V^8 = \frac{2^8}{35I}(16 - 5I),$$

and

$$E^8 = \frac{2^9}{5I}\left\{\frac{14}{16 - 5I}\right\}^7.$$

These results are shown in Figure 2.

The initial conditions are shown on the left, the derived results, on a different scale, on the right. A fall in the rate of interest from 6 per cent to 4 per cent would cause the period of production to increase from 24 to 38 months, output per head to rise by 5.45 per cent, and real wages by 4.8 per cent. My own feeling is that in this example the length of the period is more sensitive to changes in the rate of interest than it would be likely to be in actual life; and that the example, therefore, overestimates the rise in output per head and in real wages which would be brought about by a fall in the rate of interest. But, of course, that is a mere guess.

A possibly more plausible example is shown by the dotted curves, where the technical conditions $V = 2(2 - e^{-R})$ yield the functional relationship $1/I = 2 \cdot e^{-R} - (1 + R)$. A fall in the rate of interest from 6 per cent to 4 per cent would then cause a rise in the length of the period of production from 28 to 32 months, and in output per head of approximately 1.58 per cent. But to approach at all closely to realism we should no doubt have to drop the condition $\dfrac{d^2V}{dR^2} < 0$ for small values of R; and this would merely mean that, if the solution of equations (α) and (β) for a given rate of interest yielded a value of R for which $\dfrac{d^2V}{dR^2}$ was positive, then that rate of interest would be incompatible with a position of stable equilibrium.

We may now return to the analysis of our fundamental equations. Differentiating (α) with respect to R, we get

$$\frac{dV}{dR} = \frac{dE}{dR}(1 + RI) + E\left(1 + R\frac{dI}{dR}\right)$$

or, using (β),

$$\frac{dE}{dR}(1 + RI) + ER\frac{dI}{dR} = 0. \tag{δ}$$

Multiplying through by $\dfrac{dR}{dI}$, this yields

$$\frac{dE}{dI}\Big/E = -\frac{R}{1 + RI}.$$

Thus, provided that we know the actual values of R and I, we can estimate approximately the percentage change in real wages consequent

on a small change in the rate of interest, even though we do not know the actual form of V. For example if $I=5$ per cent and $R=5/2$, the unit of time being a year, a fall in the rate of interest to $4\frac{1}{2}$ per cent would cause a rise in real wages by about 10/9ths of one per cent.[15] The example above would yield a very similar result.

If we now differentiate (β) with respect to R, we get

$$\frac{d^2V}{dR^2} = I\,\frac{dE}{dR} + E\,\frac{dI}{dR}, \tag{ϵ}$$

and, writing (δ) as $\dfrac{dE}{dR} = -\dfrac{E^2R}{V}\,\dfrac{dI}{dR}$ and substituting for $\dfrac{dE}{dR}$ in (ϵ),

we have

$$\frac{d^2R}{dR^2} = E\,\frac{dI}{dR}\left(1 - \frac{ERI}{V}\right).$$

But, by (α), $\quad 1 - \dfrac{ERI}{V} = \dfrac{E}{V}$ and we have, therefore,

$$\frac{dI}{dR} = \frac{V}{E^2}\frac{d^2V}{dR^2}\ \text{or}\ I^2 \cdot \frac{V\dfrac{d^2V}{dR^2}}{\left(\dfrac{dV}{dR}\right)^2}.\ \text{by }(\beta).$$

Thus, by (γ), $\dfrac{dI}{dR}$ is negative, as we should expect.

We can now apply our analysis to the Problem of Relative Shares. For our purpose this can perhaps best be stated in the form "What is the condition that the proportion of the total product going to interest payments shall increase with the amount of time used in production?" Now the proportion of interest payments to wage payments is RI. Thus, our condition is that

$$\frac{d}{dR}\,(RI) > 0.$$

[15]

$$\frac{\delta E}{E} = -\,\frac{R \times \delta I}{1 + RI} = \frac{\dfrac{5}{2} \times \dfrac{1}{200}}{1 + \left(\dfrac{5}{2} \times \dfrac{5}{100}\right)}$$

$$= \frac{5}{450}$$

Now

$$\frac{d}{dR}(RI) = I + R\frac{dI}{dR}$$

or, using the second expression for $\frac{dI}{dR}$ above,

$$I + RI^2 \cdot \frac{V\dfrac{d^2V}{dR^2}}{\left(\dfrac{dV}{dR}\right)^2}$$

$$= \frac{I \cdot V\dfrac{d^2v}{dR^2}}{\left(\dfrac{dV}{dR}\right)^2}\left\{\frac{\left(\dfrac{dV}{dR}\right)^2}{V\dfrac{d^2V}{dR^2}} + RI\right\}.$$

Now $\dfrac{IV}{\left(\dfrac{dV}{dR}\right)^2} \cdot \dfrac{d^2V}{dR^2}$ is negative by (γ). Thus $\dfrac{d}{dR}(RI)$ will be posi-

tive if

$$\frac{\dfrac{dV}{dR}}{V} \div \frac{-\dfrac{d^2V}{dR^2}}{\dfrac{dV}{dR}} > RI. \tag{ς}$$

But $\dfrac{\dfrac{dV}{dR}}{V}$ may be described as "the rate of increase of output per head

with respect to time."

and $\dfrac{-\dfrac{d^2V}{dR^2}}{\dfrac{dV}{dR}}$ as "the rate of deceleration of output per head with respect

to time."

Thus we may word our conclusion as follows: "The share of the total
product going to interest payments will increase with the amount of

time used in production *if* the rate of increase divided by the rate of deceleration of output per head with respect to time is greater than the proportion of interest payments to wage payments." It is clear that whether this condition is fulfilled or not depends on the balance between the marginal productivity of time and the intensity with which that factor is subject to the law of diminishing returns.

By way of illustration, we may note that, if technical conditions can be represented by a function of the form $V^n = a + bR$, the proportion of interest payments will increase with the length of the period of production whatever the value of n[16]; but that, if the function is of the form $V = a - be^{-kR}$, where a, b, and k, are positive, the proportion of interest payments will decrease with the length of the period of production for values of R greater than $1/k$.

It should be instructive to compare this result with the formula given by Dr. J. R. Hicks in his *Theory of Wages*.[17] This formula states that, if two factors are combined in the production of a commodity, the proportionate share going to either will increase with the proportion of it employed, provided that

$$\frac{\dfrac{\partial x}{\partial a} \cdot \dfrac{\partial x}{\partial b}}{x \cdot \dfrac{\partial^2 x}{\partial a \partial b}} > 1,$$

where x is the quantity of the product, and a and b the quantities employed of the factors.

Let us apply this formula to our problem. Let the employed population be N, so that the quantity of the product is NV.

Then $\dfrac{\partial(NV)}{\partial N} = V$, as we have assumed a constant return to labor.

$$\frac{\partial(NV)}{\partial R} = N \frac{dV}{dR}$$

and

$$\frac{\partial^2(NV)}{\partial N \partial R} = \frac{dV}{dR}$$

---

[16] Consistent with $\dfrac{dV}{dR}$ positive and $\dfrac{d^2V}{dR^2}$ negative.

[17] J. R. Hicks, *The Theory of Wages* (London, Macmillan, 1932), pp. 117, 245.

and, therefore,

$$\frac{\dfrac{\partial(NV)}{\partial N} \cdot \dfrac{\partial(NV)}{\partial R}}{NV \cdot \dfrac{\partial^2(NV)}{\partial N \cdot \partial R}} = \frac{V \cdot N \dfrac{dV}{dR}}{NV \cdot \dfrac{dV}{dR}}$$

$$= 1$$

and so Dr. Hicks' formula would yield the result that, on the assumption of constant returns to labor, the proportions of the national dividend going to wages and interest would be independent of the length of the period of production.

We have already seen that this is not correct, and it will be worth while to consider why Dr. Hicks' formula is inapplicable to my analysis. I suggest that the reason is simple. Dr. Hicks' argument is based on the assumption that the factors enter symmetrically into the productive process, whereas it must be clear from this paper that labor and time differ not only in outward form, but also in the essential nature of their relationship to that process. It seems to me further that Dr. Hicks' formula cannot be applied to the distribution of the national dividend between labor and capital. I have no desire to enter into controversy over the possible meanings of the concept "capital," but I think that in its broad outlines the following dilemma is logically valid. We can either (a) reduce "capital" to the length of the period of production or some related construction, and then the analysis of this paper will apply, or (b) we can conceive of "capital" as an existing stock of real goods; if so, it is not homogeneous, and there exists no physical unit in which its quantity can be measured.[18] Either procedure is fatal to the line of argument which Dr. Hicks develops in Chapter VI of *The Theory of Wages*.

I shall conclude this paper by making some tentative suggestions as to the way in which an actual estimate might be made of the length of the period of production of any commodity at a given moment. Let us start by assuming that the costs of a firm can be reduced to those connected with fixed capital, raw materials and labor (including salaried workers), and consider these in turn. We have to estimate the cost of each, and the length of time before the emergence of the finished product at which it should be said to have been incurred.

1. *Fixed capital.*—The best we can do here is to take the replacement cost of the fixed capital which is actually installed, though this

[18] I believe that Dr. Hicks would now accept this statement of the position as substantially correct.

makes no allowance for obsolescence. But it does avoid difficulties concerned with changes in the general price-level. Let the length of the period of production of the fixed capital be R_F, its length of life n years, and its replacement cost F. Then on its account I shall assume, for the output of the firm in any year, that F/n was spent on labor $R_F + n/2$ years before. This is an overestimate, as some interest costs will be included in F.

2. *Raw materials.*—To avoid seasonal or speculative distortions, the average cost over a reasonably long period should be taken. A year would be enough. Let the annual expenditure on raw materials be G, and the length of their production period R_G. Then on their account I shall assume, for the output of the firm in any year, that G was spent on labor $R_G + k$ years before, where k is the number of years the raw material is in process. This again is an overestimate, as some interest costs will be included in G.

3. *Labor.*—Let W be the annual wages bill. Then we shall assume, for the output of the firm in any year, that W is spent on it $k/2$ years before it is finished. This implies that labor is expended at an even rate through time on the raw material, while it is in process, and seems as reasonable an assumption to make as any other.

Then, if R be the length of the period of production of the finished commodity, we have

$$R = \frac{\dfrac{F}{n}\left(R_F + \dfrac{n}{2}\right) + G(R_G + k) + W \cdot \dfrac{k}{2}}{\dfrac{F}{n} + G + W}.$$

This equation can easily be solved provided that we know the ratio of R to R_F and R_G. Here the proper procedure seems to me to be trial and error. Make any assumption you consider plausible, and work out from it values of R for various raw materials, producers' goods, and consumers' goods. Then revise your assumptions in the light of this information, and work out the values of R again. These second results should be tolerably accurate.

I shall make the simplest possible assumption, namely, that R, R_F, and R_G, are all equal. We then have

$$R = \frac{F + k(2G + W)}{2W}.$$

Figures for F, W, and G, for some important industries may conveniently be found in the U.S.A. Census for 1919, and I reproduce

them with the derived values of R below. As only the proportions are relevant, the units are not necessarily the same for different industries.

	F	W	G	R
Food	46	7	101	3·9
Textiles	61	15	54	2·2
Iron and Steel	87	22	48	2·1
Lumber	26	8·5	14	1·6
Leather and Leather Goods	15	4	17	2·1
Paper	24	6	13	2·1
Liquors and Beverages	78	7	22	5·7
Chemicals	56	5	37	5·9
Stone, Clay, and Glass	13	3	4	2·3
Metals (other than Iron and Steel)	18	4	4	2·3
Tobacco	60	12	48	2·7
Vehicles for land transport	24	7	25	1·9

Actually, the values of R are obtained from the formula

$$R = \frac{F + k(G + W)}{2W},$$

since the Census includes in its figure for F an allowance for the value of stock in process (kG). k has been taken as $1/12$ throughout. This figure is a plausible average over the whole of industry, but the use of an average at all is unsatisfactory when the field to be covered ranges from baking to the maturing of strong drink. But any great refinement can be seen to be superfluous when one considers that the Census has probably often obtained its figure for F by capitalizing the quasi-rents that the firm was earning; and, if so, 1919 is a peculiarly bad year to take.

However, even after all proper scepticism has been exercised, the fact that in six of the twelve industries the value obtained for R lies between $2\cdot1$ and $2\cdot3$ years remains interesting; and I hope that it may stimulate statisticians to some research on this problem which will yield more reliable results.

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DYNAMICS OF SAVING AND INVESTMENT[1]

By Edward Theiss

SAVING and investment have such an important rôle in the genesis of industrial fluctuations that it seems worth while to attempt a determinate macro-dynamic analysis of these phenomena. The exact theory in the general case would lead to very complicated mathematical problems (non-linear differential and integral equations and the like). Therefore, as a first step, it is of interest to treat certain special cases which in a simplified form still contain the essentials of reality but admit the application of comparatively simple mathematical methods. For this purpose it is best to consider the case of relatively small deviations from a stationary equilibrium, in which case the dynamic equations become linear and can be integrated by means of elementary functions.

First, we investigate saving in its relation to investment in fixed capital. We assume that there is only one type of consumption good produced and consumed. Let us denote the output of this commodity emerging per unit time at time t by $y(t)$. In order to produce this output, certain amounts of labor and capital goods (machinery, etc.) must be employed per unit time. Let these magnitudes be designated $l(t)$ and $m(t)$, respectively. These amounts to be employed at time t are determined, in general by the amount of the output and by the prices of the factors: wages and the rate of interest. In the following we shall conventionally regard the commodity or factor *labor* as the *numéraire so that wages are always equal to unity*. The rate of interest and the price of the output we denote $i(t)$ and $p(t)$, respectively, both of these being functions of time. In the treatment of fixed capital investment we shall disregard the time lag between input and output in production, so that l and m are connected with y at the same moment of time, but we do take into consideration the depreciation of capital goods, which we assume to be proportional to the output, so that its amount at time t per unit time is equal to δy, δ being a constant. Let Π be the cost of a unit of capital goods, as defined below, and P the profit per unit of output; then the gross income per unit time, due to the sale of the output, is

$$(1) \qquad yp = l + im\Pi + \delta y\Pi + yP.$$

The second term in this expression is interest cost per unit time, and the third is depreciation cost per unit time.

The cost of capital goods is composed of the costs of labor and capital goods (the latter including interest and depreciation) employed in their production. If L and M are the (constant) amounts of these factors used to produce one unit of capital goods, and if the depreciation of the capital goods in this process is given by the constant Δ, then we have:

$$(2) \qquad \Pi = L + iM\Pi + \Delta\Pi,$$

assuming here that there exists only one type of capital goods and that there is no profit in the production of capital goods. From the last equation we obtain:

$$(3) \qquad \Pi = \frac{\Lambda}{1 - \epsilon i},$$

where $\Lambda = \dfrac{L}{1-\Delta}$ and $\epsilon = \dfrac{M}{1-\Delta} = \text{constant}$.

The existence of positive profits will induce entrepreneurs to enlarge their fixed capital. The amount of *new* capital goods (in addition to replacement) necessitated by an increase per unit time of the output of consumption goods, we assume to be $\mu\dot{y}$, where μ is a known constant and y the rate of increase of the output of consumption goods. The amount, and hence the cost, of this new capital investment will be a certain function of profits:

$$(4) \qquad \mu\Pi\dot{y} = f(P).$$

This equation expresses the demand for capital due to the enlargement of production. The supply of capital consists of the savings of our community. These savings we assume to be a certain function of the income of the community only. Regarding this function we make the assumption that the percentage saved increases with the real income. In the first approximation we assume the function to be linear. In other words, if S denotes nominal savings, and R real income per unit time at t, then we put:

$$(5) \qquad \frac{S}{pR} = \rho(R - R_0),$$

where ρ is a certain constant, and R_0 the real income corresponding to the stationary equilibrium (in which there are, of course, no savings).

This equation can also be written in the form:

$$\frac{S}{p} = -\rho R_0 R + \rho R^2 = -\rho R_0 R + \rho R_0^2\left(1 + \frac{R - R_0}{R_0}\right)^2.$$

Since we consider here only small deviations from stationary equilibrium, higher powers of $\dfrac{R-R_0}{R_0}$ may be neglected and so we obtain:

(6) $$\frac{S}{p} = -\rho R_0{}^2 + \rho R_0 R = -f + gR,$$

where $f = \rho R_0{}^2$, $g = \rho R_0$.

The nominal gross income of the community is composed of (I) The proceeds yp arising from the sale of the output of consumption goods (which includes all the items in the right member of (1), and (II) The income earned by the factors employed in connection with the new investment in fixed capital (items in the right member of (2)). We have, therefore, $Rp = yp + \mu\Pi\dot{y}$.

Using this equation, we obtain from (6)

$$S = -fp + gyp + g\mu\Pi\dot{y}.$$

This can also be written in the form:

$$S = -fp_0\left(1 + \frac{p-p_0}{p_0}\right) + gy_0p_0\left(1 + \frac{y-y_0}{y_0}\right)\left(1 + \frac{p-p_0}{p_0}\right)$$
$$+ g\mu\Pi\dot{y}$$

where p_0, y_0 are the values corresponding to stationary equilibrium. If we neglect higher powers of the quantities $\dfrac{p-p_0}{p_0}$, $\dfrac{y-y_0}{y_0}$ and take into consideration that for stationary equilibrium:

$$S_0 = -fp_0 + gy_0p_0 = 0,$$

$$f = gy_0,$$

we deduce:

(7) $$S = gp_0\,(y - y_0) + g\mu\Pi\dot{y}.$$

We shall now first consider the case where the supply and demand of capital (i.e., savings and new investments) are equal. We have then:

$$\mu\Pi\dot{y} = S.$$

By means of (7) we obtain:

(8) $$\mu\Pi\,(1-g)\dot{y} = gp_0\,(y - y_0).$$

With regard to (3) we can write this equation in the form:

$$\mu \Lambda(1 - g)\dot{y} = gp_0(y - y_0)(1 - \epsilon i) = gp_0(y - y_0)$$
$$+ gp_0\epsilon y_0 i_0\left(1 + \frac{i - i_0}{i_0}\right) - g\epsilon p_0 i_0 y_0\left(1 + \frac{y - y_0}{y_0}\right)\left(1 + \frac{i - i_0}{i_0}\right).$$

Neglecting here again the higher powers of the percentage deviations of the variables from the stationary equilibrium values, we get:

$$(9) \qquad \mu \Lambda (1 - g)\,\dot{y} = gp_0(1 - \epsilon i_0)(y - y_0).$$

This equation can easily be integrated and we obtain as a result:

$$(10) \qquad y - y_0 = ce^{\lambda t},$$

where c and $\lambda > 0$ are real constants.

Under the foregoing assumptions, therefore, we have a steady increase in the output from the stationary level without fluctuations. This result is independent of the form of the demand function of the output.

If we want to determine all our unknown functions, we have to introduce the demand for the output explicitly. We assume this to be of the Evans type:

$$(11) \qquad y = -ap + b + h\dot{p},$$

where a, b, and h, are positive constants. We get by (10) a first order linear differential equation in p which has no periodic solutions. This fact can also be seen by substituting the value of y from (11) in (9). This gives

$$(12) \qquad A\ddot{p} + B\dot{p} + C(p - p_0) = 0,$$

where:

$$A = \mu(1 - g)\Lambda h,$$
$$B = -\mu(1 - g)\Lambda a - gp_0(1 - \epsilon i_0)h,$$
$$C = agp_0(1 - \epsilon i_0).$$

This second order linear differential equation has the solution:

$$p - p_0 = ce^{\lambda t},$$

where λ is one of the roots of the characteristic equation:

$$A\lambda^2 + B\lambda + C = 0.$$

These roots are always real, because the discriminant $B^2 - 4AC$ must here always be positive. This can be verified by substituting the fore-

going values of A, B, and C. The discriminant then becomes a square. The conclusion is thus independent of the values of the constant in the problem.

Next, we consider the case where the amount of savings is *not* equal to the value of new investments. We have to assume then that part of the output is hoarded. The nominal value of the hoards is by definition the difference between savings and investment, which is the part of income that is neither consumed nor invested. The volume of the hoards is changing according to some definite law supposed to be known. Regarding the function that expresses this law, we consider, similar to the procedure of Marshall,[2] Keynes,[3] and Robertson,[4] the relation of real hoards to real output or income. We might suppose that the amount of real hoards bears a constant proportion to real output or income. Another alternative more in concordance with the nature of the motives which determine hoarding is to assume that the proportion between real hoards and real output or income is decreasing with increasing output or income.

First, we consider the case when the volume (i.e., the real value) of the existing stock of hoards is always a certain fixed percentage κ of the output. Then the nominal value of the hoard H is given by

$$N = \kappa y p.$$

If we suppose again that the higher powers of the percentage deviations from the stationary equilibrium values can be neglected, we have

$$(13) \qquad H - H_0 = \kappa p_0(y - y_0) + \kappa y_0(p - p_0),$$

where H_0 signifies the hoards corresponding to stationary equilibrium (the above assumption involves that $H_0 \neq 0$).

The saving made by the community is now only partly used for new investment and the remainder increases the value of hoards. We have, therefore,

$$(14) \qquad S = \mu \Pi \dot{y} + \frac{dH}{dt}.$$

By the same approximation we used for the deduction of (9) and neglecting the terms with $\frac{i-i_0}{i_0}y$, $\frac{i-i_0}{i_0}p$, i.e., supposing that the rates of

[2] *Money, Credit, and Commerce*, p. 45.
[3] *Monetary Reform*, p. 76.
[4] *Banking Policy and the Price Level* (1932), p. 47.

change in output and prices are of the order of magnitude of the percentage deviations from the equilibrium values (this assumption is not necessary if $\epsilon = 0$) we now get:

$$gp_0(1 - \epsilon i_0)(y - y_0) = \mu\Lambda(1 - g)\ddot{y} + \kappa p_0(1 - \epsilon i_0)\dot{y} + \kappa y_0(1 - \epsilon i_0)\dot{p}.$$

Substituting the values of y from (11), we obtain:

$$(15) \qquad\qquad A_1\ddot{p} + B_1\dot{p} + C_1(p - p_0) = 0,$$

where

$$A_1 = A + \kappa(1 - \epsilon i_0)p_0 h,$$
$$B_1 = B + \kappa(1 - \epsilon i_0)(y_0 - ap_0),$$
$$C_1 = C,$$

A, B, and C, being the constants previously occurring in (12).

The solution of this equation is, similarly to that of (12), given by

$$p - p_0 = ce^{\lambda t},$$

where λ is the root of the characteristic equation

$$(16) \qquad\qquad A_1\lambda^2 + B_1\lambda + C_1 = 0.$$

The roots of this equation will be complex numbers, if the discriminant

$$D_1 = B_1{}^2 - 4A_1 C_1$$

is negative. The solution of (15) then becomes:

$$p - p_0 = ce^{\beta t}\cos(\alpha t + \omega),$$

where

$$\beta = -\frac{B_1}{2A_1},$$

$$\alpha = \frac{\sqrt{-D_1}}{2A_1},$$

and c, ω are constants of integration. We have in this case an oscillatory solution.

In order to gain the best insight into the nature of the various possible oscillatory solutions, let us consider several economies with the same p_0, y_0 and the same structural constants except a and b. D_1 is

of the second degree in a and we can determine the two values of a for which $D_1 = 0$. These values can be shown, by using the expressions for A_1, B_1, and C_1, as given above, to be always real and positive. If a_1 and a_2 denote these values, we shall have complex roots for the characteristic equation (16) and consequently oscillatory solutions of (15) when

$$a_1 < a < a_2.$$

The length of a cycle is given by

$$T = \frac{2\pi}{\alpha}.$$

The discriminant D_1 being a parabola in a, we get the shortest possible cycle if

$$a = \frac{a_1 + a_2}{2} = \bar{a}.$$

For this value of a_1 we have always $\beta > 0$. This oscillation is, therefore, explosive. If the condition:

$$\kappa > \rho h p_0,$$

is fulfilled, we can determine a value $a_3 > 0$ for which (15) has a solution corresponding to undamped oscillation. We shall have then explosive or damped oscillations depending on whether $a > a_3$ or $a < a_3$. In the first case we have an oscillatory departure from, in the second case an oscillatory approach to, the stationary equilibrium.

Assuming as a second alternative that the proportion between the volume of hoards and output is decreasing with increasing output, we can first suppose that the volume of hoards is constant. Then we have

(17) $$N - N_0 = \kappa y_0 (p - p_0).$$

Substituting this value of H into (14) we obtain an equation similar to (15) which can be treated by the same methods. As a second case we assume that the hoards increase with the output but the proportion between them decreases. In first approximation the decrease of this proportion[5] is linear and we put:

$$\frac{H}{py} = \sigma - \kappa(y - y_0),$$

[5] Another assumption leading to similar consequences is to suppose that this proportion decreases with rising prices p according to a linear function.

where σ is a constant fulfilling the condition

$$\sigma - \kappa y_0 > 0$$

which is necessary if the hoards are to increase with the output. By the same approximation as used previously, we obtain now

(18) $H - H_0 = \sigma y_0(p - p_0) + (\sigma - \kappa y_0)p_0(y - y_0).$

Substituting this expression in (14), we should deduce again an equation similar to (15) which can be discussed in the same way.

Instead of considering the relation of real hoards to output, we can make similar assumptions regarding the proportion between real hoards and real income. In this case we obtain linear differential equations of the third order in $p(t)$ if we suppose that the rates of change of $y(t)$ and $i(t)$ are of the order of magnitude of the percentage deviations from the stationary equilibrium, that is, we exclude violent and sudden changes in this variable (this restriction is not necessary, if $\epsilon = 0$). Such a third order linear differential equation may have a general solution composed of an aperiodic exponential and an oscillatory function.

We come, therefore, to the conclusion that, if there is a discrepancy between savings and the value of new investments, then, even if the time lag of production is disregarded, there is a possibility, but not a necessity, that fluctuations will arise. This conclusion depends, of course, on the type of the functions and equations assumed. In this connection the type of the demand equation is particularly important.

We now proceed to take into account the fact that production requires time. In order to avoid unessential mathematical complications, we shall assume that production does not involve the use of fixed capital (i.e., $m(t) \equiv 0$). We further suppose that the production lag is a given constant τ. In order to produce a unit of output (of consumption goods) emerging at time t, we assume that there is needed an amount γ of labor per unit time over the whole time interval from $t-\tau$ to t, γ being a structural constant. The total amount of labor needed per unit time in the interval between $t-\tau$ and t, in order to make an amount $y(t)$ emerge at t, is consequently equal to

(19) $l(t) = \gamma y(t).$

Under these assumptions we have for the value of the output:

(20) $yp = l \int_0^\tau e^{i(t-x) \cdot z} dx + yP.$

The capital required for production is now purely working capital consisting of goods in the process of production. Working capital can

be defined in a first approximation as the sum of all wages absorbed up to time t by all goods in various stages of completion present in the plants at time t. If W stands for working capital, we have:

$$(21) \qquad W = \int_0^\tau l(t + x)\,(\tau - x)dx.$$

The presence of positive profits will induce an enlargement of production requiring an increase in working capital which will, therefore, be a certain function of P:

$$(22) \qquad \frac{dW}{dt} = f(P).$$

By aid of (21) we deduce:

$$\frac{dW}{dt} = \int_0^\tau \frac{dl(t + x)}{dx}\,(\tau - x)dx = [l(t+x)(\tau - x)]_0^\tau + \int_0^\tau l(t+x)dx$$

or

$$(23) \qquad \frac{dW}{dt} = -l\tau + L,$$

where $L = L(t) = \int_0^\tau l(t+x)\,dx$ is equal to the current total wage bill. The capital required by the enlarged production we assume to be supplied by the savings of the community so that savings equal the value of new investment. If savings are determined by the same type of function as we introduced in the analysis of fixed capital investment, we shall have similarly to equation (8):

$$(1 - g)\,\frac{dW}{dt} = gp_0(y - y_0).$$

Using equation (23) we obtain:

$$(1 - g)\,L - (1 - g)\tau l = gp_0(y - y_0),$$

whence, by differentiation, we have:

$$(24) \qquad (1 - g)\,[l(t + \tau) - l(t)] - (1 - g)\tau \dot{l} = gp_0\dot{y}.$$

If we substitute the value of l from (19) in (24), we get a differential equation for $y(t)$ of a type whose characteristic solutions are well known.[6] The character of the output function is, therefore, independent

[6] R. Frisch and H. Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," this issue of ECONOMETRICA, pp. 225–239.

of the form of the demand function, since this function does not appear explicitly in (24). The explicit introduction of this function, however, is necessary if we want to determine all the unknown functions of our problem. We shall now assume this function of the simple linear form:

(25) $$y = -ap + b,$$

since it will presently appear that oscillatory solutions are now possible though the demand equation has no speculative term.

By means of (19) and (25) we deduce from (24)

(26) $$A_2 \dot{p} + B_2 [p(t + \tau) - p(t)] = 0,$$

where

$$A_2 = \gamma(1 - g)a\tau + agp_0,$$

$$B_2 = -a\gamma(1 - g).$$

The solutions of this equation are of the type:

$$p - p_0 = ce^{\lambda t},$$

where λ is one of the roots of the characteristic equation

$$A_2\lambda + B_2 e^{\lambda \tau} - B_2 = 0.$$

Since

$$\frac{A_2}{B_2} < 0, \text{ and } -\frac{A_2}{B_2\tau} > 1,$$

we have beside $\lambda = 0$ one real root $\lambda_0 > 0$ which indicates a steady deviation from the stationary equilibrium. From the infinite number of complex roots there is one:

$$\lambda_1 = \frac{\beta_1}{\tau} + \frac{\alpha_1}{\tau} i,$$

for which

$$2\pi < \alpha_1 < \frac{5}{2} \pi,$$

and the corresponding fluctuation has the longest oscillation period $\frac{4}{5}\tau < T_1 < \tau$. The oscillations are all explosive ($\beta > 0$). It can be shown that if savings are not equal to the value of new investment, i.e., in case of hoarding, there may occur oscillations with a length of period greater than the period of production τ.

After having determined $p(t)$ we can obtain also the function $i(t)$. From equation (20) we get:

$$(27) \qquad \gamma \int_0^\tau e^{i(t-x)z}\,dx = p - P.$$

We make now the simplifying assumption that we can neglect higher powers of i and (22) is of the form:

$$\eta \frac{dW}{dt} = P,$$

where η is a constant. The right side of (27), then, is a known function of t. If we differentiate (27) twice, we get an inhomogeneous differential equation of the form:

$$-\tau \frac{di(t-\tau)}{dt} + i(t) - i(t-\tau) = ce^{-\nu t}$$

where ν and c are known constants. If we make now the substitution:

$$\phi(t) = i(t)e^{\nu t},$$

then we obtain for $\phi(t)$ an equation similar in form to (27):

$$A_3\dot{\phi}(t-\tau) + B_3\phi(t) + C_3\phi(t-\tau) = 0$$

where A_3, B_3, and C_3, are known constants.

We can investigate again the influence of hoarding on the fluctuations. We shall have in case of hoarding, similarly to (14)

$$S = \frac{dW}{dt} + \frac{dH}{dt}.$$

If we assume, for example, that the proportion between real hoards and real income is a constant, we deduce the following equation for $p(t)$:

$$A_4\ddot{p}(t) + B_4\dot{p}(t) + C_4\dot{p}(t+\tau) + D_4[p(t+\tau) - p(t)] = 0,$$

where A_4, B_4, C_4, and D_4, are certain constants composed of the structural parameters. This equation may be solved by the same methods as (26).

Summarizing our results, we can state that in case of relatively small deviations from stationary equilibrium, and if we disregard the time elements of production, then, under the foregoing assumptions regarding production and savings, fluctuations may only arise if there is a discrepancy between the value of savings and investments and if the

demand function has a speculative or anticipatory term, i.e., contains at least the first differential quotient of price. If the time element of production is taken into account in its simplest form, fluctuations will be possible even though there is no discrepancy between saving and investment and the demand function has no speculative or anticipatory term. But these cycles will be shorter than the production lag. The effect of hoarding, i.e., a difference between savings and value of new investments, in this case may result in a lengthening of the duration of the cycles.

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THE CHARACTERISTIC SOLUTIONS OF A MIXED
DIFFERENCE AND DIFFERENTIAL EQUATION
OCCURRING IN ECONOMIC DYNAMICS

By Ragnar Frisch and Harald Holme

INTRODUCTION

In his lecture at the meeting of the Econometric Society at Leyden in 1933, Mr. Kalecki presented a highly interesting macrodynamic study of an economic system. His paper will appear in a later issue of Econometrica. It is seen that his analysis leads up to the mixed difference and differential equation

$$(1) \qquad \dot{y}(t) = ay(t) - cy(t - \theta),$$

where y is the unknown function (in Kalecki's notation $J(t)$), a and c non-negative constants, and θ a given (positive) lag. This type of equation is apt to occur in various kinds of dynamic economic problems. It deserves, therefore, the attention of econometricians. The present paper attempts to give a somewhat detailed discussion of its characteristic solutions.

In Kalecki's problem the constants a and c are to be empirically determined. They are equal to

$$a = \frac{m}{\theta},$$

$$(2)$$

$$c = \frac{m + n\theta}{\theta},$$

where m and n are the empirical non-negative constants discussed in Kalecki's paper. Using available statistics and smoothing the constants in such a particular way that the main cyclical solution of (1) becomes *undamped*, Kalecki obtains the values

$$(3) \qquad m = 0.95, \qquad n = 0.121, \qquad \theta = 0.6 \text{ (years)}.$$

The imposition of the condition that the solution shall be undamped is in my opinion not well founded. It is more correct, I think, to be prepared to accept any damping which the empirically determined constants will entail, and then explain the maintenance of the swings by erratic shocks. This would be an explanation along the lines indicated in my paper in the Cassel volume. If this is done, the amount of damping of the theoretical solution of (1) will be an additional feature which

may later, in a more detailed analysis, be used to explain the degree to which the observed cycles are *irregular* with respect to length. In the following, no assumption will be made, therefore, about the solution of (1) being undamped.

At the Leyden meeting a discussion arose regarding the question of whether the period $T = 10$ years obtained by Mr. Kalecki was the only solution, or whether other periods could also be found. Since the equation (1) is a mixed difference and differential equation, it is clear that its characteristic equation will be transcendental and probably give an infinity of roots. Kalecki contended, however, that all these roots would correspond to periods shorter than θ, except the one solution $T = 10$ years which he had found. More generally, he contended that there would always exist at most one solution with a period longer than θ (provided the empirical constants m and n were non-negative). He maintained that this proposition was an immediate consequence of the similar proposition which Tinbergen had developed regarding the special equation obtained from (1) by putting $a = 0$. In the discussion, the objection was raised that, even though it be true that Kalecki's equation can by a suitable transformation be reduced to Tinbergen's, at least it was not obvious why the proposition in question should hold for the solutions of Kalecki's equation simply because it holds for those of Tinbergen's. In the version of his paper, which will appear in ECON-OMETRICA, Kalecki does not go into any further analysis of this matter, but has asked me to do so. One of the results which follows from the present analysis is that Kalecki's contention is right; there actually exists at most one root giving a period longer than θ.

In the present paper the problem is treated by starting directly from equation (1) as it stands, without transforming it to the Tinbergen form. This has the advantage that the rôle played by the coefficients of equation (1) is recognized more directly.

A greater part of the mathematics involved in the present note has been worked out under my direction by Mr. Harald Holme, assistant at the University Institute of Economics, Oslo. The paper is, therefore, presented as written in joint authorship.

<div style="text-align: right">RAGNAR FRISCH</div>

I. THE CHARACTERISTIC EQUATION AND THE FUNCTION

$$f(u) = \frac{u}{tg\, u} + \text{Log} \frac{\sin u}{u}.$$

If a function of the form

(4) $$y(t) = e^{\rho t},$$

where ρ is a constant, shall satisfy (1), ρ must be a root of the characteristic equation

(5) $$\rho = a - ce^{-\rho\theta}.$$

This is verified simply by differentiating (4) and inserting into (1).

Writing ρ in the form

(6) $$\rho = \beta + i\alpha, \quad i = \sqrt{-1},$$

where α and β are real, the equation (5) takes the form

$$\beta + i\alpha = a - ce^{-\beta\theta}(\cos \alpha\theta - i \sin \alpha\theta).$$

Separating the real and imaginary parts of this equation, and putting for brevity

(7) $$u = \alpha\theta, \quad v = \beta\theta,$$

we obtain the two equations

(8) $$\frac{\sin u}{u} = \frac{1}{c\theta} e^v,$$

(9) $$\cos u = \frac{a\theta - v}{c\theta} e^v,$$

where all the quantities involved are now real.

From the two equations (8) and (9) we may eliminate v by means of elementary functions. From (8) we get indeed $v = \text{Log } c\theta + \text{Log } \sin u/u$, where Log stands for the natural logarithm. Inserting this expression for v into (9), we get

$$\cos u = \left(C - \text{Log } \frac{\sin u}{u} \right) \frac{\sin u}{u},$$

that is,

(10) $$f(u) = C,$$

where $f(u)$ is the function

(11) $$f(u) = \frac{u}{\lg u} + \text{Log } \frac{\sin u}{u}$$

and the constant C is equal to

(12) $$C = a\theta - \text{Log } c\theta.$$

It is important to notice that the function $f(u)$ may be tabulated and plotted once for all, since it does not depend on any of the empirical

coefficients a, c, and θ, which define the differential equation (1). The shape of the function $f(u)$, as well as numerical tables of its ordinate, will be given later.

Thus, in the present case the determination of the characteristic roots is reduced to the solution of *one* real equation instead of two simultaneous equations. This simplifies the problem considerably.

In order that a real exponential function shall be a solution of (1), it is necessary and sufficient that its exponent ρ be a real root of equation (5), and in order that a (damped) sine curve shall be a solution of (1), it is necessary and sufficient that its frequency α, i.e., the parameter

$$(13) \qquad \alpha = \frac{2\pi}{T}, \text{ where } T \text{ is the period,}$$

shall be $\alpha = u/\theta$, where u is a real root of (10).

<center>II. THE CRITERION FOR THE EXISTENCE OF REAL
EXPONENTIAL SOLUTIONS</center>

The equation (5) may be written

$$(14) \qquad \frac{a}{c} - \frac{w}{c\theta} = e^{-w}$$

where

$$(15) \qquad w = \rho\theta.$$

In a rectangular system of axes let us draw the two curves whose abscissa is w and whose ordinates are respectively $a/c - w/c\theta$ and e^{-w}; the first of these curves is a straight line and the second an exponential (see Figure 1). These two curves may have no, one, or two, points of intersection.

If $a > c$, the straight line crosses the ordinate axis in a point above that in which the exponential crosses the ordinate axis. Since the slope of the straight line is negative, there exist in this case always two points of intersection between the two curves of Figure 1, namely, one with a positive and one with a negative w. The case $a > c$ is, however, of no great interest in the present case; by virtue of (2) we may indeed assume

$$(16) \qquad a < c.$$

This being so, let us first determine the condition that must be fulfilled in order that the two curves in Figure 1 shall have exactly one

point in common when $a<c$. Since the slope of the straight line is negative, this is equivalent to determining the condition for tangency between the two curves. The slope of the straight line is $-1/c\theta$ and that of the exponential is $-e^{-w}$, the condition for tangency consequently $e^{-w}=1/c\theta$, that is,

$$(17) \qquad w = \text{Log } c\theta.$$

Inserting this in (14), we obtain the equation $C=1$. Thus, if there is tangency between the two curves, we must have $C=1$. The reciprocal
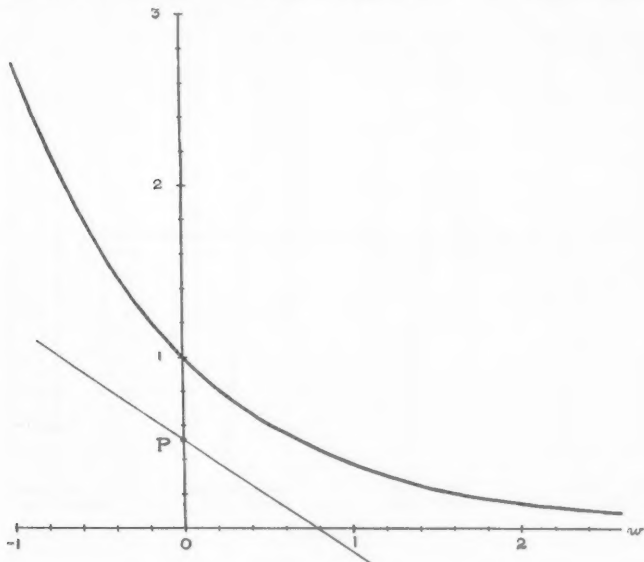
FIGURE 1

is also true. Indeed, suppose that $C=1$ and let us consider the points on the two curves that have the common abscissa $w=\text{Log } c\theta$. It is easily found that these points must also have the same ordinate on the two curves and the same slope, consequently, it must be a point of tangency; $C=1$ is, therefore, the necessary and sufficient condition for tangency. Obviously, at most one point of tangency can exist. Such a point of tangency means that (14) has two coinciding solutions, which in turn means that the two time functions $e^{wt/\theta}$ and $te^{wt/\theta}$ are both solutions of the differential equation (1), w being the abscissa of the point of tangency in Figure 1.

Now suppose that $C > 1$. By keeping c and θ constant and *decreasing* a—if necessary down to minus infinity—it will always be possible to reduce C from its original value, which was above unity, to some value below unity. Since C is continuous in a, it must consequently pass $C = 1$. But reducing a, while c and θ are kept constant, means *lowering* the point P in Figure 1 where the straight line crosses the ordinate axis, but keeping the slope of the straight line unchanged. In other words, if $C > 1$, we get tangency by lowering P and keeping the

$$F(u) = \frac{u}{tg\,u} + \text{Log}\,\frac{\sin u}{u}$$
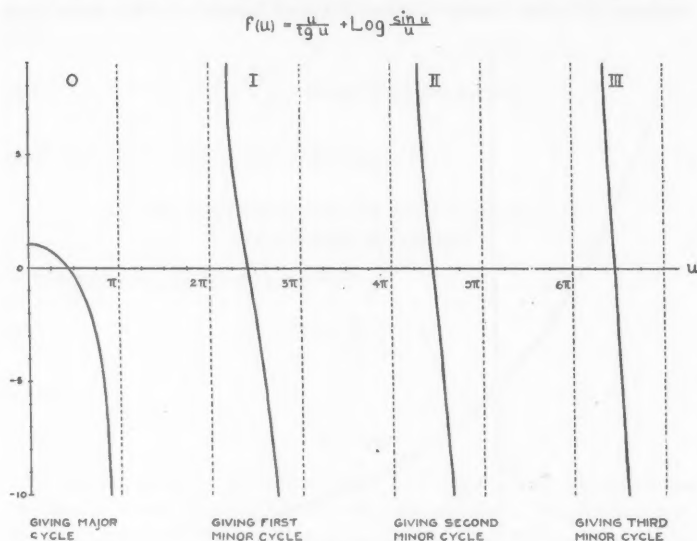
FIGURE 2

slope unchanged. This means that in the original position the straight line must have *intersected* the exponential. Since the exponential is always curved upwards and extends to infinity in both directions, exactly *two* points of intersection must have existed in the original position.

On the other hand, if originally $C < 1$, it is always possible by increasing a, and keeping c and θ constant, to bring C up to 1. It is even possible to do this without letting a surpass c. Indeed, for $a = c$, C becomes equal to $c\theta - \text{Log}\,c\theta$ and it is well known that this difference is always larger than unity when $c\theta$ is positive. Thus, if $C < 1$, we get into a situation of tangency by *raising* P, which means that originally the straight line did *not* intersect the exponential.

This shows that (if $a < c$) *the equation* (1) *has no, one, or two (different) real exponential solutions, according as* $C < 1$, $C = 1$, *or* $C > 1$. In the case $C = 1$, there further exists a solution which is t times an exponential.

TABLE 1.—TABLE OF $f(u) = (u/tg\ u) + \text{Log}\ (\sin u/u)$.

0. $0 \leqq u \leqq \pi$		I. $2\pi \leqq u \leqq 3\pi$		II. $4\pi \leqq u \leqq 5\pi$		III. $6\pi \leqq u \leqq 7\pi$	
u/π	$f(u)$	u/π	$f(u)$	u/π	$f(u)$	u/π	$f(u)$
0.	1.0000	2.00	∞	4.00	∞	6.00	∞
0.02	0.9979	2.125	13.26				
0.04	0.9920	2.25	4.766	4.30	6.999	6.30	11.18
0.06	0.9822						
0.08	0.9682	2.37	1.128	4.37	3.235	6.41	2.807
0.10	0.9504	2.38	0.8754			6.42	2.141
0.12	0.9283	2.39	0.6268				
0.14	0.9024	2.40	0.3793	4.40	1.815	6.43	1.485
0.16	0.8718	2.41	0.1346	4.41	1.356	6.44	0.8350
0.18	0.8372	2.42	−0.1088	4.42	0.9016	6.45	0.1884
0.20	0.7981	2.43	−0.3508	4.43	0.4529	6.46	−0.4529
0.22	0.7546	2.44	−0.5920	4.44	0.0081	6.47	−1.096
0.24	0.7063	2.45	−0.8340	4.45	−0.4357	6.48	−2.952
0.26	0.6533	2.46	−1.076	4.46	−0.8767		
0.28	0.5953					6.6	−9.819
0.30	0.5320	2.625	−5.605	4.5	−2.649	7.0	− ∞
0.32	0.4636	2.75	−11.14	4.6	−7.416		
0.34	0.3891	3.00	− ∞				
0.36	0.3090						
0.38	0.2226						
0.40	0.1298						
0.42	0.0296						
0.44	−0.0780						
0.46	−0.1934						
0.48	−0.3179						
0.50	−0.4516						
0.60	−1.296						
0.70	−2.598						
0.80	−4.912						
0.90	−10.91						
1.00	− ∞						

NOTE.—The inaccuracy of the above values of $f(u)$ amounts to about 1 or 2 units of the last digit given.

III. THE CRITERION FOR EXISTENCE OF CYCLES

To each real solution in u of the equation (10) corresponds a cycle in the form of a sine curve that may either be damped, undamped, or "explosive." The existing solutions are immediately indicated by plotting the function $f(u)$ and noticing the points where this curve intersects the horizontal drawn at the height C. Figure 2 and Table 1 indicate the shape of the function $f(u)$.

Since $f(u)$ is an even function in the argument u, it is sufficient to study it for positive u. This is only another expression for the fact that imaginary roots of the characteristic equation must always occur in conjugate pairs. This is seen by noticing that the left members of (8) and (9) are even functions of u, so that, if a given u satisfies the characteristic equation, the same u with its sign reversed (and with the same v) must also satisfy.

From Figure 2 it is seen that no real branch of the function $f(u)$ exists over the intervals from π to 2π, from 3π to 4π, from 5π to 6π. etc., while in each of the other intervals a branch exists that decreases monotonically from $+\infty$ to $-\infty$, except in the interval from 0 to π where the branch only extends from $+1$ to $-\infty$. This latter branch of the curve is also given in a larger scale in Figure 3.

This indicates immediately what sorts of cycles will exist for a given C. If $C < 1$, exactly one root u will exist in each of the intervals

$$(18) \qquad\qquad 2k\pi \leqq u \leqq (2k+1)\pi. \qquad (k = 0, 1, 2 \cdots)$$

If $C > 1$, no root will exist in the first of these intervals while exactly one root will still exist in all the others. This means that if $C < 1$ the equation (1) will have exactly one cyclical solution in each of the period ranges

$$(19) \qquad\qquad \frac{\theta}{k+\frac{1}{2}} \leqq T \leqq \frac{\theta}{k} \qquad (k = 0, 1, 2 \cdots)$$

$$(T = \text{period})$$

while if $C > 1$ it will have no solution in the first of these ranges (i.e., for $k = 0$), but exactly one solution in all the other ranges. In the period ranges $\dfrac{\theta}{k} < T < \dfrac{\theta}{k-\frac{1}{2}}$ $(k = 1, 2, \cdots)$, there can never occur any solution.

The solution in the first range, i.e., for

$$(20) \qquad\qquad 2\theta \leqq T \leqq \infty,$$

we shall call the major cycle, the others we shall call the minor cycles. All the latter are shorter than the lag θ, while the first is larger than θ. From Figure 2 it is even seen that it is larger than 2θ.

In the limiting case $C = 1$, we may still say that (10) has a solution, but it is a degenerate one, namely $u = 0$, which means that the corresponding cycle has an infinite period; in other words, it is nothing but an exponential. Thus, we retrieve here the result of Section 2, namely, that in the case $C = 1$ there exists an exponential solution.

The various possible cases are summarized in Table 2.

TABLE 2.—NATURE OF THE SOLUTIONS OF EQUATION (1), WHEN $a \leqq c$.

	Exponentials	Major cycle (with a period longer than the lag θ)	Minor cycles (with periods shorter than the lag θ)
$C > 1$	Two exponentials with different exponents.	No major cycle	Exactly one minor cycle in each of the period ranges $$\frac{\theta}{k+\frac{1}{2}} \leqq T \leqq \frac{\theta}{k}$$ $(k = 1, 2, 3, \cdots)$ No cycles in the period ranges $$\frac{\theta}{k} < T < \frac{\theta}{k-\frac{1}{2}}$$ $(k = 1, 2, 3, \cdots)$.
$C = 1$	One pure exponential and one exponential multiplied by t (time). The exponents in these two time functions are equal.	No major cycle	
$C < 1$	No exponential	Exactly one major cycle. Its period must be equal to or larger than 2θ.	

IV. CALCULATION OF THE ROOTS

Each of the real roots, if they exist, may be determined to a first approximation w_1 by a graphical reading from Figure 1 and then improved to a second approximation w_2 either by the formula

(21a) $$w_2 = \theta a - \theta c e^{-w_1}$$

or by

(21b) $$w_2 = \text{Log } c\theta - \text{Log } (a\theta - w_1).$$

If necessary, the process can be repeated, inserting each time in the second member the new approximation found for w.

The formula (21a) is obtained by writing (14) in the form $w = g(w)$, where $g(w) = \theta a - \theta c e^{-w}$, and then replacing w in the left member by w_2 and in the right member by w_1. Similarly (21b) is obtained by writing (14) in the form $w = g^{-1}(w)$, where g^{-1} is the inverse function of g. In other words, (21b) may be looked upon as obtained by solving (21a) with respect to w_1 and then interchanging the subscripts 1 and 2.

It is well known that the condition for convergency of an iteration process based on an equation of the form $w = g(w)$ is that[1] the absolute value of the derivative $g'(w)$ be less than unity over the interval where

[1] C. Runge, *Praxis der Gleichungen*, Goschens Lehrbücherei, (1921), p. 48.

the iteration takes place. Since the derivative of the inverse function is one divided by the derivative of the function itself, it will in some cases be best to use (21a) and in others (21b). In practical work a few tentative calculations will soon indicate which one of the two procedures is the best.

The convergency of the process will as a rule be greatly improved by using the powerful formula of Steffensen[2]

$$(22) \qquad w' = w_1 - \frac{(\Delta w_1)^2}{\Delta^2 w_1}.$$

Here, w' is the improved value of w; w_1, w_2, and w_3, being three successive approximations found by the original process, that is, in this case by equations (21a) or (21b). The difference symbols in (22) are defined by

$$(23) \qquad \Delta w_1 = w_2 - w_1,$$
$$(24) \qquad \Delta^2 w_1 = w_3 - 2w_2 + w_1.$$

If a still greater accuracy is desired, the process defined by (22) may, of course, be iterated. The formula of Steffensen is in many cases capable of producing convergency even though the original process does not converge.

For the cyclical components, the parameters u and v can be determined as follows. If the frequency is not very small, one starts by determining a first approximation to u by a graphical reading from Figure 2 (or, for the major cycle, better from Figure 3). The corresponding value of v is determined from the equation

$$(25) \qquad v = \mathrm{Log}\left(c\theta\,\frac{\sin u}{u}\right)$$

which is only another form in which to write (8). Inserting the value of v thus determined into the right member of (9), we get a new value of u, which again may be inserted in the right member of (25). Continuing in this way, closer and closer approximations to the numbers u and v will usually be obtained. This is the method that has been used for the numerical computations in Tables 3, 4, and 5. An alternative procedure would be first to determine the solution of the one variable equation (10) to the accuracy desired, using a similar method to that indicated in (21a), (21b), and (22), and then finally determining v by (25).

If the root u in question is small, the graphical reading from Figure 3 becomes difficult. In this case one may start by finding an approximation to v. For this purpose we consider the formula

[2] *Skandinavisk Aktuarietidskrift* 1933, p. 68. See also the paper by Harald Holme, *ibid.* 1932, p. 229.

$$(26) \qquad\qquad v = a\theta - \frac{u}{tg\,u},$$

which is obtained by eliminating e^v between (8) and (9). When C is close to unity, u for the major cycle is very small and, hence, $\dfrac{u}{tg\,u}$ close
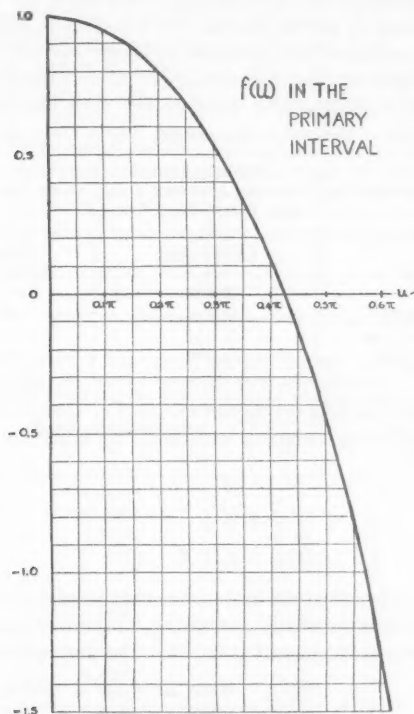
FIGURE 3

to unity. Consequently, if C is close to unity, v will be approximately equal to

$$(27) \qquad\qquad v = a\theta - 1.$$

Incidentally, this is the value of v that maximizes the right member of (9).

When a first approximation to v is determined by (27), (9) and (25) may be used alternately as before to determine closer and closer approximation to the set (u, v) in question.

In most practical cases the parameter C will probably lie between 0 and 1, which by the criteria of Section 3 means that all the solutions are cycles. If we assume that C lies in this interval, it is possible to indicate a much sharper lower limit for the period of the major cycle than the one given in (20), and it is further possible to indicate a very narrow range for each of the minor periods. Since, as is seen from Figure 2, the function $f(u)$ is monotonically decreasing, the limits in question are obtained by putting in (10) $C=0$ and $C=1$, respectively. The ranges thus determined become more and more narrow, as we consider shorter and shorter cycles. The successive curve-branches to the right in Figure 2 indeed pass zero (and unity) with greater and greater steepness. Table 3 indicates the period limits thus obtained for the

TABLE 3.—IF $0 < C < 1$ (AND CONSEQUENTLY ALL SOLUTIONS CYCLICAL) THE RATIOS BETWEEN THE PERIODS AND THE LAG θ MUST LIE BETWEEN THE FOLLOWING LIMITS

	Lower limit	Upper limit
Major cycle	4.706	∞
First minor cycle	0.828	0.843
Second minor cycle	0.450	0.453
Third minor cycle	0.310	0.311

major cycle and the first few minor cycles. For the still shorter cycles, the periods will even be given with good accuracy by the following limit

$$(28) \qquad \frac{\theta}{k + \frac{1}{4}} < T \qquad \text{(when } 0 < C\text{)}$$

$$(k = 0, 1, 2 \cdots).$$

In order to prove this, we first notice that in each of the possibility intervals $f(u)$ is monotonically decreasing. This is obvious from Figure 2 and may be proved in general as follows. The derivative of $f(u)$ is

$$(29) \qquad f'(u) = \frac{df(u)}{du} = \frac{u \sin 2u - (u^2 + \sin^2 u)}{u \sin^2 u}.$$

We are only considering the intervals $2k\pi < u < (2k+1)\pi$; here the denominator of (29) is positive and the numerator is negative for $u > 0$. Indeed, the function

$$(30) \qquad \phi(u) = u \sin 2u - (u^2 + \sin^2 u)$$

is zero for $u = 0$, and its derivative

$$(31) \qquad \phi'(u) = 2u (\cos 2u - 1) = - 4u \sin^2 u$$

is negative for $u > 0$.

This being so, let us consider the value of $f(u)$ in the *middle* of each of the possibility intervals, that is, for $u = (2k + \frac{1}{2})\pi$. Here we have

(32)
$$f = - \text{Log } (2k + \tfrac{1}{2})\pi,$$

which is negative for all non-negative k. Thus, the point where $f(u)$ vanishes must be situated in the *left* half of each of the intervals. This in connection with Figure 2 shows that the point where $f(u)$ vanishes approaches from the left towards the midpoint of the interval as we consider intervals situated more and more to the right.

The goodness of the approximation (28) is illustrated by comparing it with the lower limits for the first few cycles given in Table 3.

TABLE 4

	Exact lower limit when $0 < C$	Lower limit by (28)
Major cycle	4.706	4.000
First minor cycle	0.828	0.800
Second minor cycle	0.450	0.444
Third minor cycle	0.310	0.308

TABLE 5.—THE PERIOD T AND THE DAMPING FACTOR AS FUNCTIONS OF THE LAG θ WHEN $m = 0.95$ AND $n = 0.121$.

Lag θ	Empirical constant C	Period of major cycle T	Damping factor pr. period $e^{2\pi\beta/\alpha}$
0.01	1.0000	∞	0
0.012	0.9997	3.08	0.0000
0.02	0.9987	2.56	0.0019
0.05	0.9949	3.14	0.0526
0.1	0.9886	4.21	0.1656
0.2	0.9761	5.76	0.4288
0.4	0.9516	8.10	0.7016
0.6	0.9277	9.95	0.9725
0.8	0.9043	11.54	1.224
1.0	0.8814	12.98	1.461
1.4	0.8372	15.56	1.925
2.0	0.7744	18.95	2.609
3.0	0.6777	23.90	3.768
4.0	0.5895	28.38	4.968
5.0	0.5085	32.57	6.241
6.0	0.4336	36.57	7.584
7.0	0.3639	40.38	8.962
8.0	0.2987	44.15	10.43
9.0	0.2376	47.78	11.94
10.0	0.1799	51.35	13.49

V. THE PERIODS AND THE DAMPING EXPONENTS AS
FUNCTIONS OF THE EMPIRICAL CONSTANTS

In his first paper, Kalecki gives some examples of how the length of the major period changes with changes in some of his empirical constants, under the assumption that the constants must always be such that the major cycle turns out to be undamped. For the reasons previously given, it seems to be more significant to ask how the length of the major cycle and the damping exponent will both vary when cer-
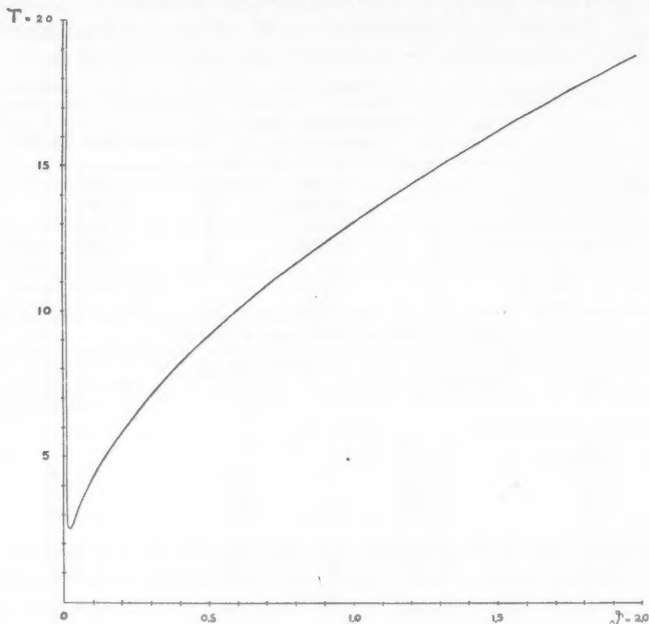
FIGURE 4

tain unconditional changes are made in the empirical constants. In particular, it is interesting to see what changes will be entailed if Kalecki's constants $m = 0.95$ and $n = 0.121$ are maintained unchanged but the lag θ varied. The result is as indicated in Table 5 and Figure 4.

It is seen that the period, roughly speaking, changes proportionately to θ, until θ comes down to about 0.02. From here on the period shoots up almost perpendicularly, if θ becomes still smaller.

It will further be noticed that there is actually a damping of the major cycle when θ is less than approximately 0.6, that is, the value

adopted by Kalecki. And there is "anti-damping" when θ is larger. We only need to make a slight change in θ from Kalecki's value in order to have an "anti-damping" which is absolutely unrealistic. This is another fact which seems to point out that there is something artificial in fixing, by convention, the values of the constants in such a way as just to obtain an undamped solution.

The variation of the minor periods with θ is described rather closely already by the limits indicated in Table 4 and by formula (28). It is seen that all these periods change virtually proportionally to θ, no matter how the empirical constants are determined, provided only that $0 < C < 1$; the first and second minor periods are, for instance, always about 0.83 and 0.45, respectively, times the lag θ.

University Institute of Economics, Oslo

AVIS DE LA RÉUNION DE LA SOCIÉTÉ D'ÉCONOMÉTRIE À NAMUR (BELGIQUE), SEPTEMBRE 1934

La cinquième réunion européenne de la Société internationale d'Économétrie se tiendra á Namur (Belgique) du 23 au 26 septembre 1935.

Le programme des travaux, qui n'est pas arrêté, sera conçu dans le même esprit que ceux des précédentes réunions.

Les membres qui envisagent de participer á la réunion sont priés de le faire connaître dès que possible à M. le Professeur Léon H. Dupriez, 15, Avenue du roi Albert, Louvain (Belgique), qui a bien voulu se charger de recueillir les adhésions et leur fera parvenir les circulaires ultérieures.

Les adhésions devront être accompagnées du versement de 25 francs belges (5 belgas) pour la couverture des frais d'organisation (par chèque de banque ou par virement postal au compte de chèques postaux n° 139934).

Les membres désireux de proposer une communication devront en adresser le texte, ou tout au moins une analyse assez détailée, le plus tôt possible, et en tout cas avant le premier juillet prochain, au Président du Comité d'Organisation, Professor A. L. Bowley, London School of Economics, Houghton Street, London.

ANNUAL SURVEY: SUGGESTIONS ON QUANTITATIVE BUSINESS CYCLE THEORY

By J. Tinbergen

CONTENTS

SECTION I. INTRODUCTION

1. The aim of business cycle theory is to explain certain movements of economic variables. Therefore, the basic question to be answered is in what ways movements of variables may be generated. In answering this question it is useful to distinguish between *exogen* and *endogen* movements, the former being movements during which certain data vary, while, in the latter, the data are supposed to be constant. Since in a *static* theory (e.g., the Walrasian system) the values of all the variables are determined by the data at the same moment, this theory does not admit endogen movements. A dynamic theory is therefore necessary, a theory being called "dynamic" when variables relating to different moments appear in one equation. Of course, exogen movements are possible in a dynamic theory as well as in a static one. Business cycle research shows that a dynamic theory is necessary to explain facts and so business cycle theory is not possible within the field of static theory. We have to pay attention, therefore, to a number of fundamental dynamic notions. The first thing is to distinguish between the "mechanism" and "exterior influences." The mechanism is the system of relations existing between the variables; at least one of

these relations must be dynamic.[1] This system of relations defines the structure of the economic community to be considered in our theory. Such a mechanism may perform certain kinds of swinging movements that are characteristic of the system as such ("Eigenschwingungen"), and, under certain circumstances, it may show other fluctuations whose character is dependent, to a smaller or larger extent, on the nature of the exterior influences, the "impulses."[2] Frequently, the impulses present themselves as given initial conditions of the variables—comparable with the shock generating a movement of a pendulum—or as given changes of the data entering the equations. Examples of such shocks will be given later.

It is important to distinguish between the *mathematical form* of the mechanism, i.e., of the equations defining it, and the *economic sense* of these equations. The mathematical form determines the nature of the possible movements, the economic sense being of no importance here. Thus, two different economic systems obeying, however, the same types of equations may show exactly the same movements. But, it is evident that for all other questions the economic significance of the equations is of first importance and no theory can be accepted whose economic significance is not clear.

From the foregoing, it is evident that all business cycle theories may be looked upon as generalizations of static theory in the sense that one or more "spots" in the system of static equations describing the economic community are "dynamized." In the next section I shall give a survey of the most important dynamic relations existing in reality, paying special attention to those which have been more or less verified by statistical investigations. I shall then treat certain of the recent attempts which give—with the help of rigorous simplifications—a closed system of equations, a closed system being one that contains just as many variables as equations. As a rule, the analytical form of the equations is simplified as much as possible, otherwise no explicit solution would ever be possible. Such a system admits of one or more solutions and leads to a definite movement, initial conditions being given. This deserves the name of a business cycle theory, although it may be a very much simplified one, but "open" systems cannot represent theories, since they do not give a complete system of hypotheses sufficient to determine the movement of the variables. Many literary

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[1] In the next chapter a large number of examples of dynamic relations between economic variables will be given. The above definition of statics and dynamics falls into line with the one given by Frisch, *Nationaløkonomisk Tidsskrift*, 1929.

[2] The term "impulse" is here taken to cover both discontinuous and continuous changes (cf. 25).

theories seem to fall in this category, in that they do not clearly state all relations which are necessary or are tacitly included. Examples will be given. I have attempted also to discuss some of the generally known recent theories from this point of view. The use of mathematics is of peculiar value in this field.

SECTION II. THE FACTS

2. In this section a survey will be given of the most important dynamic relations existing in real economic life which may, or must, be chosen as starting points for an adequate business-cycle theory. It is evident that in principle there is a very large number of dynamizations. In almost every part of the economic system reality shows divergencies from static theory. It is virtually impossible to give them all their places in any given "closed" theory. Some simplifications must, therefore, be made and, as in all applied sciences, the question is how to find the happy medium between the complexity of the real world and the simplicity of an amenable theory. Suppose that we confront a complete static system with its demand and supply equations, budget equations, monetary equations, and technical and other institutional equations. We must ask how to group the subjects, the markets, the institutions, etc. We must inquire which dynamizations must be chosen as the more important ones.

The procedure in this section will be as follows: I shall give a certain highly simplified grouping of the elements of economic life. The principal remaining relations will be discussed and special attention will be paid to their dynamic features. In particular, those dynamic features which have been verified statistically, or that seem to possess a high degree of evidence, will be treated. But the number of relations enunciated is still too large to make an explicit solution possible. In the next sections, therefore, further simplified systems are given and partially solved. It will be possible, in the end and to a certain degree, to combine such solvable cases. The solutions of the simple cases to be discussed will, however, themselves exhibit remarkable features.

The grouping of the elements, which has its statistical counterpart in the calculation of index numbers of all sorts, is characteristic of what Frisch calls *macrodynamic* economics in contrast with *microdynamic* economics; of course, this subdivision also fits the static type. In the macrodynamic field, I propose the following grouping of subjects, objects, and markets:

Subjects: 1. Producers:
            Entrepreneurs: A. Bankers
                           B. Others

                          Wage earners
                          Capitalists
                          Government
                    2. Consumers⎫ being largely the same
                    3. Savers   ⎬ subjects as before

The usual reservations must be made; certain individuals may be both extrepreneurs and capitalists, both wage earners and capitalists, both consumers and savers.

        Objects: 1. Original factors of production:
                      Nature
                      Labor
                      Capital Disposal
                        Short term
                        Long term
                    2. Products: Raw Materials—for Consumables
                                              —for Means of Production
                                  Finished Products—Consumables
                                                   —Means of Production
                    3. Property rights: Stocks and shares

Markets: Generally speaking, a market may be defined for each of these objects. Depending on the viewpoint of the investigator, the market for the production factor, "nature," may be the market for hiring land or, when agriculture is placed outside the "cycle generating community," the agricultural raw materials' market.

We have now to sum up which groups of subjects contribute to the supply or the demand in each of the groups of markets and also how these supply and demand contributions behave in dependence on the variables adopted. Within certain limits, this choice of variables is free. We can state *a priori*, however, that categories remaining constant, or nearly constant, throughout a business cycle should not be taken as variables. It is, on the contrary, a natural simplification of our problem that we may consider such categories as constants.[3] Variables may be, e.g., the general price level, or subdivisions of it, wage rates, profit margins, the quantity of investment goods produced or commanded, money rates, etc. Further, it is preferable *to measure all quantities from their long term equilibrium value as a base*, i.e., in deviations from this equilibrium value.

3. In consequence of the very important rôle which the decisions of entrepreneurs play in different markets, it is useful to begin with an analysis of the activities of these subjects. They form the basis for the

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[3] This procedure is analogous to Ehrenfest's method of "adiabatic variations" in theoretical physics.

supply of products as well as for the demand for production factors. The following subdivision of these activities seems to be reasonable. There are activities relating to: (A) current production, (B) replacing of worn-out means of production, (C) new investment, and (D) speculation.

Current production.—Current production is carried on with the help of existing equipment, which may or may not be used fully. The capacity of equipment itself is a result of past increases and decreases, which, in general, depend on earlier values of variables such as profit margins, etc. The total capacity of existing equipment will, therefore, have the character of an integral over time of these margins, the integral being possibly taken with a certain weight function.

The influence of total *existing* capacity on production does not seem to be important in the few cases where information is available, except in the case of shipping. One of the reasons might be the small deviations of capacity from its trend, which may be considered as its long term equilibrium value.[4]

The second factor determining output, viz., the percentage of capacity in use, shows much wider variations. This percentage will depend, roughly speaking, on the profit expectancies at the moment in which the decision to produce is taken. This means that there is a time lag between these profit expectancies and the quantity delivered.

Profit expectancies do not necessarily coincide with profits earned at the moment of decision nor with those really earned on the production planned at that moment. There is, however, a tendency to correspondence between profits earned at the moment of decision and profit expectancies, as the former are an important factor in the evaluation of the latter.

Discrepancies arise, e.g., out of psychological influences such as general confidence or lack of confidence and out of facts indicating an impending change. Such facts may be of an exogen character, as inventions, crops, etc., or they may be of an endogen character, as the value of price velocities dp/dt, p being the price level and t time.

Profits earned depend on the method of calculation and the special values of the entities included in that method of calculation. As to the method, it may be remembered that costs of materials may be taken as relating to the moment of delivery or to the moment of purchase of the individual materials used. In the latter case another lag is introduced.

The number and character of entities included in the calculation of profits are influenced in the first place by the "horizon" of the entrepreneur, i.e., the period of time over which he is calculating his profits.

[4] Cf., however, 12.

This may sound theoretical. Each case of "distress selling," however, to which an important place in business cycle theory is given by Fisher and others, represents an example of a certain shortening of the entrepreneur's (in this case the dealer's) horizon. This case is so complicated that a separate treatment seems necessary; I can only mention it here. In certain cases it may be formulated in Goudriaan's way: below a certain price level the supply curve is negatively sloped. I doubt, however, whether, in a first approximation to the business cycle problem, these negatively sloped supply curves should be introduced. The frequency of this phenomenon seems not to be such that for production as a whole any increase of production during the heavy price fall of 1929–1932 can be observed. A simpler example of the influence of horizon is given below (8, raw material markets).

The most important categories included in the profit calculation are prices and quantities of products and production factors.

As to prices, the most variable ones included are, in general, the prices of products and raw materials,[5] which may be combined in the price margin. In mineral and agricultural production particularly, the behavior of price margins is important for the course of profits and, therefore, for that of production.[6] For industry this relation has not been investigated sufficiently. It is possible, and for the post war period even probable, that the rigid way in which prices of industrial products are formed has weakened this relation, where it existed before.[7] On the other hand, Mitchell attaches great importance to the course of prices as a cause of the movement of production. The great difficulty in statistical research in this field is the scarcity of finished products prices for the pre-war period. Thus far, our knowledge of the influence of prices on production is meagre.

Other prices appearing in the profit calculation are those of the factors of production. Among those the first is wages. My impression is that the influence of wages on the cyclical movements of profit margins has been exaggerated and that this is partly due to political controversies. Nobody will deny, of course, that the price of labor has some influence. There are two things, however, that tend to reduce this

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[5] Raw materials are factors of production, when we consider industry (and finance, trade, etc.) apart from agriculture. Raw materials are, then, so to say, the contributions of nature to the system. The choice of this more restricted system rests on good grounds, as cyclical movements are particularly industrial and financial phenomena; agriculture only shows them irregularly.

[6] Hanau, "Die Prognose der Schweinepreise," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 18. Berlin, 193; Bean, "The Farmer's Response to Price," *Journal of Farm Economics* (1929), p. 368; Tinbergen, "De Wisselwerking tuschen loon en werkgelegenheid," *De Ned. Conjunctuur* (September 1933), p. 10.

[7] Wagemann, *Konjunkturlehre* (Berlin, 1928), p. 174.

influence on *cyclical* movements of production below what is generally assumed. First, the variability of wages is much smaller than that of prices. In accounting for changes in profit margins, prices are, therefore, far more important. Second, at least for enterprises already existing, only in variable costs are the wages responsible for changes in production, as constant costs do not influence it. Of course, constant costs may be very important for the level around which profits are oscillating and, therefore, for the welfare of entrepreneurs. For the cyclical variations of production volume they are, however, only of secondary interest.[8]

Perhaps these two circumstances explain the statistical fact that the influence of wage rates on production cannot be shown clearly.[9]

The prices of the second principal factor of production, *i.e.*, money rates, also seem to have only a small influence on profit margins for current production. Speaking of money rates, we should distinguish between short term rates and long term rates. As far as can be shown statistically, both seem to have only a very small influence on the variations of production occurring at the same time. The chief reasons for this fact may perhaps be expressed as follows: the total amount of short term interest included in costs of production is small, whereas the variations in the amount of long term interest are small. This is rarely contested as to current production in its limited sense. It is, however, contested as to trade and building activity, to both of which the above mentioned facts do not apply. In accordance with this, pre-war building activity in Germany[10] shows cycles fairly contrary to general business cycles. For England and the United States, however, building cycles are concurrent with general cycles. As to trading, the influence of short money rates has not yet been shown statistically. Thus, the influence of money rates on current production and trading volume should also be investigated more exactly.[10a]

[8] They may be of some interest in so far as they influence the moment at which a failure may occur and this failure may lead to reorganization. Wages also influence the demand side of the finished product's market; this will be mentioned later on.

[9] Cf. my paper mentioned in note 6, parts of which are reproduced in Methorst and Tinbergen, "Les recherches relatives à la conjuncture au Bureau Central de Statistique des Pays-Bas," *Revue de l'Institut Intern. de Stat.* I (1934), p. 37. As is stated there, in my opinion the figures given by Rueff on this problem, and discussed at the Tokio (1930) Session of the Institute, indicate that the chief influence on the price-wage relation originates from prices. (Cf. *Bull. de l'Inst. Int. de Stat.*, Tome XXV, 3e livr., p. 765.)

[10] Cf. Hunscha, "Die Dynamik des Baumarkts," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 17, Berlin 1930.

[10a] After I had finished this survey, C. F. Roos' admirable paper on building activity in St. Louis was published in his *Dynamic Economics*, Bloomington, Ind.,

In so far as long term money rates influence current production volume, this influence may have a considerable lag; in the profit calculation money rates of long ago may appear, owing to long term loans.

Apart from prices, different "*quantities*" have influence upon profits. The most important example in our present line of argument (cf. p. 249) is the quantity of production factors required for a unit of product, the productivity. It is evident, for example, that an increase in productivity, other things being equal, raises profits. In some cases the influence of productivity on quantity produced has been proved statistically to be important.[11] Changes in productivity themselves may be of a different nature. Distinction must be made between sporadic changes caused by important inventions and systematic changes caused by systematic improvement of methods used. Of course the limits between those two cases are not clearly defined but the distinction seems practical. The first category is especially emphasized by Schumpeter (his "new combinations"). The second category may be distinguished again in two sub-categories, automatic and non-automatic changes. By automatic changes, I mean changes caused by the variations of quantities produced, leading automatically to variations in overhead labor per unit of product. By non-automatic are meant all other systematic changes. An example of a non-automatic systematic change may be indicated. An examination of the statistical material available shows that, in different branches, a deviation of total production of minus one per cent from its trend is followed after about half a year by an increase of productivity per man of one per cent per annum above its trend value.[12]

These variations in productivity are related to a phenomenon emphasized by Hayek, i.e., *variations in roundabout way*. The Böhmian notion of a roundabout way may be considered as a means of measuring the method of production used. Assuming that technical knowledge remains unchanged, there exists a one-dimensional range of possible methods of production. Each of these methods is, in general, characterized by (a) a certain length of roundabout way, (b) a certain degree of mechanization, and (c) a certain productivity. Supposing, on the other hand, each special (a) or each (b) or each (c) represented by only one possible method, it would be indifferent whether we used (a) or (b) or (c) as a measure for that method, and there might be the

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1934; here the very distinct and large influence of the credit position (the number of foreclosures) was proved to exist.

[11] Cf. the papers mentioned in note 7.

[12] Cf. Tinbergen, "De invloed van de conjunctuur op de arbeidsproductiviteit," *De Nederlandsche Conjunctuur* (June 1934), p. 13.

possibility that Hayek's phenomenon and ours would be one and the same. There is, however, doubt as to the validity of both hypotheses just mentioned. There is a large possibility that the length of round-about way varies only very slowly and continuously, and then it has no significance for business cycle theory. It is very difficult, if not impossible, to get statistical evidence on these questions, and for that reason it is doubtful whether this phenomenon should be mentioned at all in this section, dealing as it does with factual evidence.

Up to this point we have assumed, in accordance with Walrasian theory, that the entrepreneur fixes his output in response to the price situation, independent of the quantity demanded. It is even characteristic of the Walrasian line of argument that, when there is inequality of demand and supply, prices will change first of all, and it is only these changed prices which, in turn, induce the producer to change his supply. We may formulate this by saying that the initiative in producing is taken by the producer, who bases his activity on prices, etc. For large sections of the economic community, however, another form of "market organization" exists, that is, one in which the initiative is taken by the buyers, and where the determining factor for production is the order, or in which the initiative is still taken by the producer, who bases his acts on expected orders rather than on prices. Modern budget methods, in fact, work in that manner.[13] It may be observed that, in these circumstances, the producer is no longer a real "entrepreneur," but only a representative of his buyers. I make no objection to this observation; the principal thing is that our equations will be different. This phenomenon of "passive entrepreneurs" presents itself in a certain degree when production is carried out "on order" and not "for stock." It must be recognized, however, that the investing activity of entrepreneurs will be "passive" only in a very small number of cases.

I have now terminated the description of the more important factors which in an "observable" way influence profits and, therefore, the amount of current production. A further question concerns the quantitative relation between profits and the quantity produced. In general,

[13] The fact that, especially in recent business cycles, prices show a lag behind production does not of necessity prove that, on the average, prices are not the determining factors. They may be, for other factors may be present, showing a lead before production, such that the proper combination of prices and these factors shows a small lead also. As examples of these factors the following may be mentioned: (a) the price derivative with respect to time (changing velocity of prices) and (b) labor costs of production, moving *inversely* parallel to the integral of price-deviations from trend over time. A more serious indication against what is called here the "Walrasian" version is the poor correlation existing between profit margins and production for separate *industries*.

we can assume that the larger the profits, the larger the quantity produced. It seems, however, more exact to formulate the relation as follows: the amount produced depends on that part of the total capacity for which profitable exploitation is possible or, still more exactly, for which exploitation is more profitable than idleness. The precise form of this relation depends on the distribution of the different classes of profitability. This distribution is, in many cases, close to a Gaussian one.[14] But, for small variations, a linear approximation may be used. In the following sections, many examples of this approximation will be discussed.

I have already stated that, in general, a certain period of time will elapse between the decision to produce and the emergence of the product. This is especially true of investment goods. For ships this time lag amounts to about a year, for most machinery a quarter to half a year, for large buildings and roads even longer. In the production of consumables a lag of some months also exists, its length depending on the stages included.[15] It is an important question for statistical research whether this lag should perhaps include the growing of the agricultural raw materials, representing on the average perhaps half a year, or the planning of investment works. The answer depends on the exact form of the theory considered. But, in several cases there is good reason for not including them. This is especially true when there are surplus stocks of raw materials sufficient for the larger volume of production required. As soon as the volume required exceeds these possibilities, a larger lag becomes necessary, and thus the lag must, at least for this reason, increase during the upward movement of business. It is interesting, in this respect, to remark that, during the downward movement, the shortest lag possible will be prevalent, since raw material stocks will be sufficient all the time.

It should be remarked, in addition, that for cases in which only the lag for the industrial process, and not that for the agricultural process, is taken into account, the corresponding profit margin, too, should be taken for industry only (cf. 13, 15).

Apart from the total length of time production required, the shape of the activity distribution curve of the process is important. By this

[14] Cf. various investigations of the Bureau of Labor Statistics (e.g., nos. 411, 412, 441, 474, 475) which, although they do not give exactly what is needed here, make it seem probable that the distribution under discussion is a Gaussian one.

[15] For some data computed from Dutch production statistics, see *De Ned Conj.* (August, 1934), p. 32. These data relate chiefly to metal industries; data on residential building are added. Only one branch, ship building, shows production periods of over one year. Iron-construction and machinery show figures of 3 to 8 months.

I mean the function indicating how many units of factors of production are to be used per time unit and per unit of product at each moment between the beginning and the end of the process.[16] This function is of a certain importance—at least in principle—for the amount and the "timing" of the buying power of the factors used.

4. Replacing of Worn-Out Capacity.—What is said in the following relates essentially to such means of production as have a "long" duration of life. Although the limit between "long" and "short" is not sharp, the distinction has great importance. It should be chosen in the neighborhood of the lag of production. For some purposes the "very long living" means of production may be considered separately. Among these we reckon all means lasting more than, say, 15 years.

Generally speaking, the history of an individual machine or other long living means of production will be characterized by decreasing productivity. There will come a moment, therefore, at which it no longer pays to use that machine, depending not only on the velocity with which productivity decreases but also on the economic situation in general. When the moment has come, one of two things may occur. The machine may be replaced by a new one or it may not. Probably a large part of worn-out machinery will be replaced. Besides, other new machines will be installed. This activity is not discussed here but under "New Investment."

To begin with, we consider a very much simplified case in which the following conditions are fulfilled:[17]

(a) all worn-out machines are replaced by new ones,

(b) the moment of replacement is independent of the economic situation,

(c) there is only one sort of machine, and

(d) there are no individual or random differences.

The consequence of (d) is that there is, for each machine, a fixed duration of "life," that of (c) and (d) that this life time is the same for all machinery. In consequence of (a), the number of machines replaced will be equal to the number worn out, which will be the number "born" at the moment just a life time before.

It is evident that the four premises mentioned are not fulfilled in reality. By replacing them by more complicated hypotheses, we gradually approximate real phenomena. Assumption (a) is perhaps the least unrealistic. It might be changed slightly by assuming that there

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[16] Cf. Frisch, "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel*, London, 1933, p. 181, 182 ("advancement function").

[17] This theoretical set-up has been considered by the Norwegian economist Dr. Kr. Schönheyder, *Statsokonomisk Tidsskrift*.

is a certain proportion between machines worn out and machines replaced.[18] It is evident that, by so assuming, we loosen the dependence existing between replacement and total demand for machinery; the smaller our proportion, the larger the part of total demand consisting of new investment—assuming total increase in production capacity is given. This is only possible when there is a certain surplus capacity, for then immediate replacement is not necessary.

A similar less stringent dependence between demand for new machinery and its construction one life time before presents itself when assumption (b) is generalized and the moment of replacement is supposed to depend on the economic situation. In these two cases, the generation of a new boom will depend in a less marked degree on the life time of the machinery and in a larger degree on other cyclical tendencies in the system, or on external (exogenous) changes in productivity.

Another generalization is introduced by changing assumption (c) and assuming that there are different sorts of machines, each sort having its own life time. Then there will be—supposing now, for simplicity's sake, that (a) and (b) again hold—a category of machine being replaced every three years, another every four years, and so on.

A somewhat different scheme develops when, instead of assumption (c), we generalize assumption (d) by considering the possibility of random differences in life time. The result is, then, that at the same "spot" in the economic system a series of different "life periods" is observed.

Statistical information on these questions is scarce. Only a few "life time distributions" of certain machines and transportation equipment are available[19] and they all show a wide spread. Apart from that, the divergences in average life time between different sorts of means of production are large.[20] Moreover, recent investigations on the volume of replacement in comparison with that of new investment show that—at least for Germany since 1924—new investment oscillates very much more widely than replacement.

New Investment.—I now pass to the activity that, at present, is looked on as the most important one in connection with business cycles, new investment activity.

In general, it may be said that decisions to invest will be based on considerations very similar to those leading to current production.

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[18] The "coefficient of upkeep," Frisch, mimeographed lectures.

[19] Cf. data for automobiles in *Quarterly Journal of Economics*, Feb. 1933, and for railway rolling stock, "De levensduur van spoorwegmaterieel," *De Ned Conjunctuur*, March, 1934, p. 22.

[20] A list is given by Vos, *De Socialistische Gids* (1932), p. 861.

When profit expectancies are large, more will be produced and invested; when they are smaller, less will be invested and produced. It is not quite the same profit calculations, however, which induce more or less investment and lead to more or less product. Some of the cost elements, for example, which are fixed for an already existing enterprise are variable for planning.[21] Something analogous may be said of the financing of new investment especially. Although, in general, capital will not be taken at fixed interest, and so in the strict sense long term interest is not an important cost element, at least for big enterprises the condition of the capital market is generally considered as of importance to the success of financing. For current production, however, the condition of the capital market is not of great importance.

The parallelism just mentioned leads to a parallelism in the production of consumers' goods and capital goods.[22] In fact, however, it is only a tendency that may be approximated all the better when there is a surplus capacity. When capacity is fully utilized, there may be a technical impossibility of such parallelism. It is then necessary that the increase in current production be preceded by an increase in capacity. Writing $x(t)$ for current production (current demand), and supposing that at every moment capacity is fully used, while all elements of capacity are equally productive, capacity must be $cx(t)$, where c is a constant. It follows that new investment is given[23] by $c\dot{x}(t)$, which forms another sort of tendency.[24] If the movement is cyclical, new investment should *lead* current production (current demand) by about a quarter period.

These formulae are related to the widespread opinion that "production of investment goods shows the business cycle earlier than production of consumers' goods." The evidence available is not, however, uniformly affirmative. An inspection of the chief sources shows the following situation.

[21] Cf. Schneider, *Theorie der Produktion*, Vienna, 1934, pp. 27–29.

[22] Although it does not exactly relate to entrepreneurial activity, a factor may be mentioned here that tends to strengthen this parallelism, viz., the spending of wages by workers, both in consumers' goods and capital goods industries, chiefly on consumers' goods. So, when capital goods industries are very active, this will be a factor for an almost unlagged corresponding activity of consumers' goods industries and conversely when activity is low.

[23] Cf. the discussion between Clark and Frisch in the *Journal of Political Economy*, 1931–32.

[24] As long as cyclical movements occur, the two relations mentioned are incompatible in their rigid form. It is only possible to combine them, therefore, when the one or the other is "generalized" by the introduction of more variables, as prices of consumers' goods, idle capacity, etc. (cf. 13).

Capital issues, on which Mitchell chiefly rests, show, indeed, for the period 1890–1913, a (not very clear) lead of about one year relative to general business activity (employment, prices, etc.). For the post-war period, this seems not to be the case. The correlation is not high and it is improved by adding some forms of short capital supplied. The improved series show, however, a lead of only about 3 months. On the other hand, J. M. Clark's data on car loadings and construction of new cars (*Overhead Costs* p. 395) did show a connection between new investment and $x(t)$ which, for the $3\frac{1}{2}$ year cycle, corresponds to new investment, leading about one year.

Indexes of production of investment goods and of consumers' goods, (for Germany[25] and for England[26] before the war and many countries after the war) do not always show a very good correlation in pre-war periods. So far as correlation exists (and for post-war periods it is even high) no definite lag in either direction is to be found. There are pre-war business cycles for which production of investment goods leads and others of which the opposite is true. All this is also valid for Swedish figures concerning employment in the two groups over a long pre-war period (see Figure 1).

Statistics relating to individual industries, such as railways,[27] shipping, cotton spinning, and blast furnaces, show divergent results.

In this connection, a knowledge of production periods in these industries is useful. Some figures are available for Holland[28] showing average lags of no more than 6 months for iron construction, machinery and other metal branches, except ship building, which shows one year and more.

In a recent correspondence between the present author and Professor Frisch on these seemingly diverging data, the latter suggested the explanation that the Mitchell-Clark relation is a relation between consumption, x, and "capital-starting," y, while the data of Figure 1 show the connection between consumption, x, and the "carry-on-activity," z. It is of course, quite conceivable that y moves approximately synchronously with x (the Mitchell-Clark relation) and, at the same time, z moves approximately synchronously with x. According to Frisch, further experimental calculations on erratic-shock-maintained swings of the systems discussed in his paper "Propagation Problems and Impulse Problems . . . " actually give movements where both the above kinds of synchronisms are present.

[25] Cf. "Die Industriewirtschaft," *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 31, 1933.

[26] Cf. W. Hoffmann, "Ein Index der industriellen Produktion für Grossbritannien seit dem 18 Jahrhundert," *Weltw. Archiv.*, Sept. 1934, p. 383.

[27] Cf. Clark, *The Economics of Overhead Cost*.

[28] Cf. *De Ned. Conjunctuur*, August 1934, p. 32.

Scarcely any statistical analyses as to the determining factors of new investments have been made. The relative influence of economic categories (prices, wages, money rates, "general confidence" etc.,) and of technical ones (exogenous factors such as new inventions, etc.) is not, therefore, known with any exactitude. An inspection of English capital issues by branches of industry suggests that the influence of general factors is rather small. There is need of further analysis along these lines.

5. *The Labor Market.*—As has already been stated, the demand for labor is largely determined by the decisions of entrepreneurs to invest
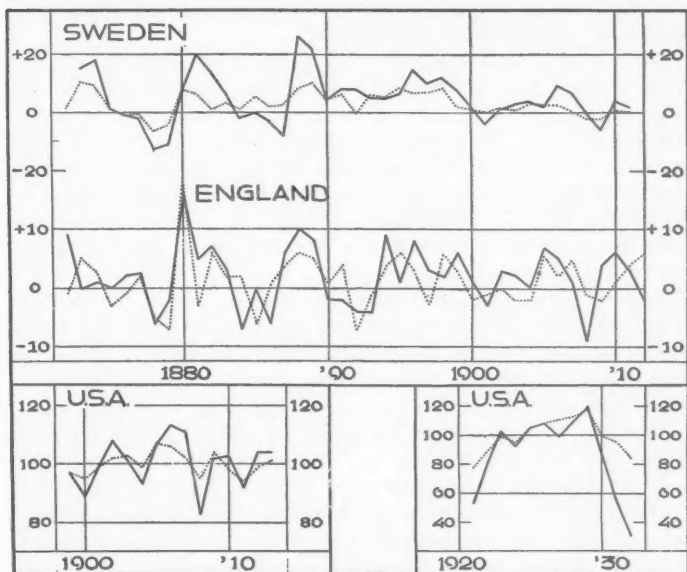
FIGURE 1.—Activity in consumers' goods (——) and capital goods industries (——). Sweden: Percentage increase of workers occupied. England: Percentage increase of production (Hoffmann, *Weltw. Archiv,* 40 (1934), 383). U.S.A., pre-war: Trend percentages of production. U.S.A. post-war: production indexes.

or to produce non-investment goods and, therefore, as a first approximation, by profit calculations. These may manifest themselves after certain lags, depending on the period of production and other characteristics of the production process. One of the determining factors is the wage level, which, in turn, is determined in the market now under discussion.

Granting free competition, the question of how employment is determined would be rather simple. Supply would be given by the number of workers present and would show no cyclical variations.

The assumption of free competition, however, has no value as an explanation of the movements of the labor market, since under its aegis no unemployment is possible. This does not mean, however, that the problem of interpreting wage movements is solved by stating that wage formation is monopolistic. For even an ideal monopolistic fixation of wages would also mean an immediate adaptation to varying circumstances, i.e., a specific statical reaction. The reaction is, however, a dynamical one that may be formulated, as a first approximamation, as a simple lag of, say, one year.

There is also another possible way to interpret wage movements. Wage changes may be assumed to be inversely proportional to the excess unemployment, i.e., to the excess of unemployment over a certain normal level. The strength of underbidding will grow with such an excess; the power to *increase* wages will grow as unemployment is smaller. Wages themselves will, then, move inversely proportional to the integral curve of excess unemployment, i.e., almost proportional to the integral curve of excess profits. For sine curves, this will simply mean a lagging behind profits but with a lag of two years (one-fourth of the period of the eight year cycle). For non-sine curves, the lag may be smaller, especially for real business cycles; apart from that there may be assumed a second influence on wages, viz., prices. Combining these two influences, one finds one year lags also possible, as prices move about simultaneously with profits. Statistical investigation undertaken by the present author shows for different countries and periods that this second explanation agrees slightly better with the facts; the difference in correlation coefficients is, however, very small.

A third interpretation is given by Mitnitzky,[29] a synthesis of a static and a dynamic relation. Basing his argument on carefully examined statistical material, he concludes that during the upward movement there is no lag between employment and wages whereas during the downward movement a lag exists. Since Mitnitzky does not choose any exact mathematical relation to represent his theory, a comparison with other interpretations is not easy; moreover, his relation is essentially of a character that cannot be expressed simply in mathematical terms. For this last reason, it is not suitable for use in a simple mathematical scheme of cycles, but for more detailed analysis his theoretical set-up seems to correspond well with the facts.[30]

[29] "Lohn und Konjunktur vor dem Kriege," *Archiv. f. Sozialw. u. Sozialpol.* LXVIII (1932), p. 318.

[30] There is, however, a disagreement with a fact pointed out by Keynes,

Apart from the regularities just mentioned, the influence of large strikes or lockouts, such as in England in 1926, should be mentioned. These must be considered as exogenous influences.

The amplitude of the cyclical wage movements is rather small, as shown by available data. As a first approximation to a more complicated theoretical scheme, therefore, wages may, without violation of the facts, be supposed to be constant (i.e., to have only a trend movement).

6. *The Credit Markets.*—Quantitative dynamic investigations into the formation of prices in the credit markets are not numerous.[31] In the face of the great importance attached by many theorists to credit problems in the understanding of business cycles, this is remarkable. Moreover, it is regrettable that static theories on interest seldom pay much attention to a fact which in reality is of great significance, viz., the existence of different markets for long and short credits. These markets behave so differently that only in an extremely rough theoretical consideration may they be treated together.

First a few general remarks.

a. We consider the market for new credits only. For short credits this does not differ much from all credits outstanding; for long term credits, of course, it does.

b. New capital issues belong to this subject only in so far as they concern fixed interest securities. Stock issues, at least formally, are not credit transactions, and, besides, a market cannot be considered if no price is contracted.

c. A third remark relates to a very general quantitative aspect of credit problems. A chief function of credit is to provide for the differences between expenses and income. The amount of new credits necessary at a given rate of interest is given by this difference. This is essentially the fundamental idea introduced by Knut Wicksell and later utilized by Keynes and others. The movements of this difference and, therefore, of credit demand at a given rate of interest may be quite different, depending on the tendencies that govern the movements of expenses and incomes of individuals and, above all, of enterprises. Figure 2 illustrates this schematically. Assuming expenses and incomes to move synchronously, but expenses to show a larger amplitude, new

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Treatise on Money, I, 303, where it is stated that: "after a large proportion of the unemployed factors have been absorbed into employment, the entrepreneurs . . . will begin to offer higher rates of remuneration."

[31] Cf., however, "Money Rates and Loan-Deposit Ratio" ("demand-supply ratio"), Persons, *Review of Econ. Stat.*, 1924, p. 266. Interrelations between "deposits" and "loans and discounts," and their ratio, U. S. National Banks outside New York, adjusted figures, Persons, *loc. cit.*, 1924, p. 281.

credits will move synchronously also (case I). Assuming that expenses and incomes show the same amplitude, but a phase difference (case II), new credits show a still larger phase difference. It is clear that there is a wide variety of other possibilities.

  d. Credit markets are especially susceptible to variations of "confidence." Certain events may suddenly shock confidence in state fi-
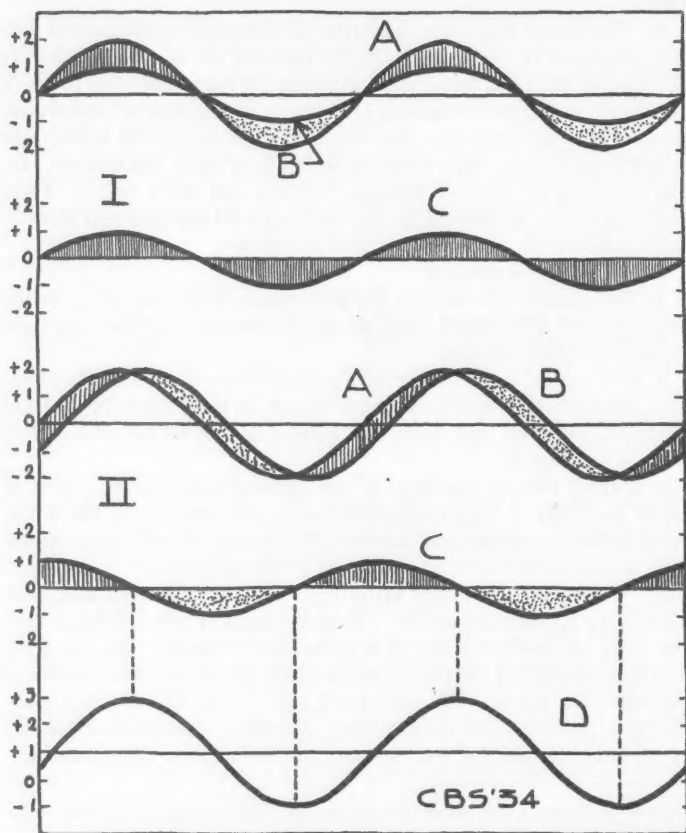
Figure 2.—Schematic representation of credit demand (at fixed interest rate). C, as difference between expenses A and income B. Case I: synchronous movements of A and B, A's amplitude surpassing B's. Case II: equal amplitudes of A and B; B lagging behind A. D: total credit amount demanded since beginning of operations (method of Bolza).

nances or in the economic future in general, and, in consequence, interest rates will rise (e.g., the autumn of 1931).

e. Interrelations between short and long term credit markets are to be sought particularly on the supply side. Saving may, in many cases, as readily take the form of supplying short as long term credit. Apart from prices in both markets, general aspects of the economic future (the "degree of confidence") may influence decisions in one direction or the other. A systematic investigation into these interrelations by Lorenz[32] showed that they would be best described in the following dynamic way: month to month changes of long term rates correlate highly with the margin between short and long term rates. The results of this interesting investigation have not yet been sufficiently utilized in further analyses; it seems that especially the causal nexus: short interest rate—long rate—confidence—investment—general activity, might be investigated more extensively.

The total supply appearing in credit markets (and to be distributed over their divisions in the way just indicated) originates from several sources: (1) Income of private persons not spent or hoarded or invested directly; (2) Temporarily idle funds of enterprises (these will, of course, supply the short credit markets particularly); and (3) Credits created by banks.

Different sorts of dynamic relations are introduced by the fact that the amounts mentioned under (1) and (2) often originate from earlier periods. Exact investigations at these points are difficult, as many statistical data are lacking. The elasticity of the monetary system resulting from the existence of the third source has been discussed so fully by competent authors that I need not go into detail here. It may be remarked again, however, that statistical information is far from ideal on this point. It should be remembered that a larger or smaller *elasticity* of the credit supply does not form a dynamic element in our sense but is an important static feature of the system.

Turning now to discussion of the short term credit market (the *money market*), we first notice that this market shows a highly variable price. In a first approximation this must be ascribed to varying demand.[33] Demand may be separated into three categories: (1) Demand for stock speculation; (2) Demand for business; (3) Demand by governments, etc. Of these categories, the first and third one seem to show

[32] Eine Differentialgleichung der Wirtschaftsforschung und ihr Integral. *Blätter für Vers.- Mathematik und verwandte Gebiete*, Beilage zur Zeitschrift für die gesamte Vers.-Wissenschaft, Bd. 29, Heft 3, p. 212.

[33] Cf. *De Ned. Conj.* Aug. 1934, pp. 78 ff. where it is stated that, except in the case of New York before the war, the influence of supply could not be traced.

only a very slight sensitiveness to price, i.e., to money rates.[34] The second category must be divided into categories according to liquidity. "Advances" (England) and "all other loans" (U.S.A.) do not show a distinct sensitiveness to rates, but bills discounted (except treasury bills) do (cf. Figure 3). They form, however, only a very small portion of total credits outstanding and, therefore, it seems logical to say that not much "regulation of demand" by money rates takes place.
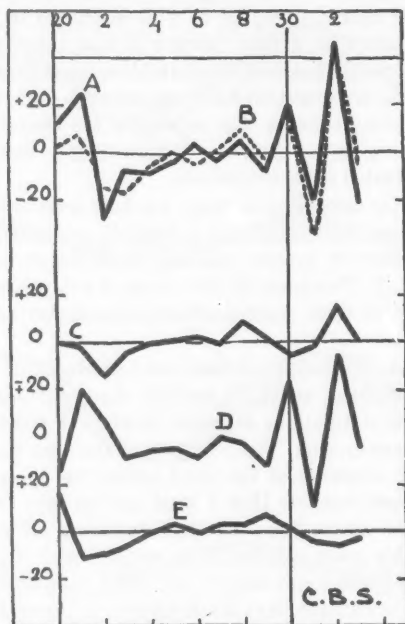
FIGURE 3.—Annual percentage change in bills discounted by English joint stock banks, actual (A) and calculated (B) from:
  C. influence of annual percentage change in treasury bill circulation;
  D. influence of change in money rate, and
  E. influence of activity, measured by total value of foreign trade (*De Ned. Conj.*, Aug. 1934, p. 24).

The conclusions given here depend, however, in some degree on what is a good index of the demand for credit at a given rate, for only when such an index is known can the relative influence of money rates and other factors on demand be estimated by correlation analysis.

[34] Cf. Owens and Hardy, *Interest Rates and Stock Speculation* (Brookings Institution); *De Ned. Conj.* Aug. 1934, p. 18.

A few words may be added on the factors that govern demand, apart from money rates. Demand for stock speculation shows, of course, high correspondence with stock sales at the exchange and with stock prices, demand by governments with deficits on their services. Advances to business show the well-known and interesting feature of maxima early in each business cycle; they are replaced, in later phases of the cycle, by capital issues. Neither increase in advances nor issues shows a high correlation with business activity, but their sum does.

Still less can be said of the long term credit markets (capital markets). The most striking feature is the large constancy of long term money rates throughout long periods. Quantitative researches into demand and supply of new capital (bonds) have not been made.

7. *The bond and stock markets.*—Closely connected with the market for new bonds is that for "old" ones; in general, it may be said that they move parallel and, theoretically speaking, may be considered to form one market. Looking at this total market, other supply and demand sources join those mentioned above. Supply of ("old") bonds (or demand for the corresponding credit, to maintain our previous terminology) is yielded by speculators or by those who must liquidate some reserves; demand comes from savers or speculators, both terms being used in the widest sense. Quantitative analyses from this point of view (e.g., considering the factors that induce an enterprise to liquidate reserves or to build up reserves) have not yet been made.

In contrast with new stock issues, dealing in old stocks can be said to have a market, although not in the sense valid for bonds, for, whereas bond yields are to a large extent known beforehand, stock yields are not.

Price formation of stocks has been investigated by Donner.[35] His investigation shows that prices are governed, as might be expected, chiefly by profit figures. As a smaller investigation made by the author suggested,[36] movements of stock prices are not proportional to profits, but show a smaller amplitude. In years of high profits the possibility of reactions, and in years of low profits that of improvement, is discounted.

Prices on the stock exchange seem to be of importance for the course of business in two ways. First, they seem more or less to regulate the amount of new capital issues, and, second, they form a basis for credits to, and consumption by, speculators, and so influence demand for commodities. Exact investigations are not yet at our disposal but it may be mentioned that the second influence, if strong enough, would

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[35] O. Donner, "Unternehmungsertrag und Zins als Faktoren des Aktienmarkts," *Weltw. Archiv*, XL (1934), 116.

[36] *De Ned. Conj.* Dec. 1932, p. 12. ECONOMETRICA, July 1933, p. 247.

be important also for the fact that the consumption of speculators would depend on their profits, i.e., on the velocity of the rise of stock prices (or on the velocity of decline, in the case of a bear speculator).

8. *Markets for consumables.*—The principal aspects of the supply side are discussed above under the heading "entrepreneurial activity." It may be emphasized that the exogeneous influence of crops is large, especially for consumables.

As to the *demand* side, the most important point seems to be that the great bulk of consumers save only a very small and stable part of their incomes. The remaining part is spent independent of the height of the price level. Of course, the distribution over different articles is not independent of the prices of these articles and of the general price level. But, with a high degree of approximation, one may say that the volume of goods and services bought by these consumers is determined simply by the amount of their incomes and the general price level, i.e., the cost of living.[37] So the amount of income is the most important determining factor. This depends, in turn, to a large degree, on employment and, therefore, on profit calculations of entrepreneurs. As already mentioned, dynamic elements may enter here. Employment shows a "distributed lag" after profit expectancies, and wages per hour probably depend on a time integral over excess profits (cf. 5), etc. There may be another lag between income payment and income spending, depending on the so-called income period.[38] I doubt, however, whether this lag has much influence since it is small compared with the other lags entering in the problem.

Apart from this great bulk of consumers, there are, of course, others for whom the distribution of their income between saving and spending seems to be more variable. These are small in number but their incomes are high and, especially for more elastic consumptions, their demand will be of importance. As a first approximation, the part saved will largely depend on profit expectancies.

Perhaps pure speculators' income is most interesting in this respect. Since, for the speculator, the income will depend on the rate of increase of stock prices, it will reach a maximum before stock prices themselves. Fluctuations in these incomes will be very large and, even though they form only a moderate part of total income, the fluctuations will be of importance for total income fluctuations.[39] As far as

[37] Robert M. Walsh, "Empirical Tests for Price Theories, Fisher, Foster and Catchings, Keynes," *Qu. J. of Econ.* XLVIII (1934), 546.

[38] This income period is given an important role by Koopmans, "Zum Problem des 'Neutralen' Geldes"; *Beiträge zur Geldtheorie, herausgegeben von F. A. Hayek* (Vienna, 1933).

[39] This point is stressed by Limperg especially.

income statistics allow a judgment, this statement seems to hold for the United States in the post-war period.

Raw Material Markets.—A consideration of raw materials markets apart from finished products markets has several purposes. If we consider agriculture as a field outside the real stage of business cycles—a defensible view-point—raw material markets are to be considered as the exponents of some of the "irregular shocks" working on the system, i.e., the shocks caused by varying crops. Whether or not we adhere to this view-point, the distinction of raw material from finished product markets is, of course, an expression for the existence of a vertical division of labor. Although vertical division of labor does not add principally new elements in our search for dynamic relations, it has, as will be shown further on, quantitative influence. In this connection, it should also be noticed that raw material markets and markets for finished goods differ essentially on the point of stock holding, this being a dynamic feature par excellence.

Stocks of finished products offer the less interesting features of the two. As far as statistical experience goes, they are, roughly speaking, proportional to production one year before and are small compared with production.[40] Raw material stocks seem much more important and seem, too, to be theoretically interesting. In an earlier issue of this journal,[41] I tried to sketch the essence of the mechanism by which these stocks generate. To recapitulate briefly—and, therefore, perhaps not exactly—, just after a crop, producers are in the possession of a supply consisting of this crop plus the carry-over from the previous year. They expect a certain average crop and demand for other years, depending on price. They distribute their supply over these different years in such a way as to have a maximum of (partly expected) profit. The demand expectancies may depend not only on general business conditions at the moment, but also on their velocity of movement.

9. *Markets for instrumental goods.*—The supply side of this market may be considered a special case of entrepreneurial activity as to current production. The special features of this market are (a) a large part of production is production on order; (b) the period of production is rather long and reaches, in ship building, for example, 1–2 years, and (c) a large proportion of cost is overhead cost.

The demand side is governed by the replacement and new-investment activities of entrepreneurs. An important difference between this market and that for consumables is the fact that purchases are not limited to the income or receipts of the buyers. To effectuate the purchase of instrumental goods, credit may be attracted and, therefore,

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[40] *De Ned. Conj.* March 1933, p. 77.
[41] ECONOMETRICA I (1933), 247.

profit expectancies and probably technical circumstances such as wear-and-tear are predominant.

For the market as a whole, demand may be considered predominant, an assumption in accordance with the known positive correlation between cyclical changes in prices and production of pig iron and other metals.

The Circulation of Money.—Some authors draw special attention to the amount of money in circulation and the velocity of circulation. These factors do not, however, constitute new elements apart from production, prices, and new credits. Whether a higher sales volume is completed with more money or with money circulating more rapidly seems of only secondary interest to the business cycle theorist. Experience shows that both things happen more or less in proportion, and thus velocity of circulation is a practical symptom of business activity.[42] Whether it is a necessary link in the casual chain of events, or whether the introduction of the notion more readily explains the business cycle, seems open to doubt.

SECTION III. RECENT THEORIES

10. *The non- and semi-mathematical theories.*—In section II, I have summed up a number of dynamic relations entering a system of equations when one tries to approximate reality closer than by static equilibrium equations. In the present section I propose to discuss some of the recent theoretical schemes. I start with a discussion of the general structure of these schemes and the quantitative assumptions on which they are based. I shall discuss the mathematical equations of the schemes, whether or not these equations are explicitly given. In a further section, the solutions of these equations and the conclusions based on them will be treated; in those sections the common features of most schemes are placed in the centre. In the present section the individual features will demand our chief attention. I will begin with two rather well known systems, Hayek's and Keynes'.

Both are examples of "open" systems. The number of variables referred to in the discussion is larger than the relations that are precisely and explicitly stated. An attempt at a mathematical "translation" leads the reader to many unsolved questions. This is not to deny that many original and valuable suggestions are made and, by stating the various assumptions and relations with mathematical precision, one may contribute to a better understanding of the unsolved problems.

As to Hayek's theory, I see as its central thought the statement that changes in the price system not only lead to changes in quantities

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[42] Snyder, *Review of Econ. Stat.*, 1925, p. 256.

produced but also in the method of production, such as the degree of mechanization. The question is legitimate whether these changes are really important. I have the feeling—and I admit it is a feeling—that technique changes continuously only, and not cyclically. We need figures.

Suppose, however, that Hayek is right. Then, I venture to doubt that the Böhmian notion of "roundabout way" is a suitable measure of "method used" in these problems (cf. 4). It should not be forgotten that Böhm uses this notion especially for static equilibrium problems, or perhaps for slowly changing equilibria.[43] The notion of "roundabout way" is less useful for the analysis of rapidly changing situations. For the simple case used as an example by Hayek which, in principle, consists of a production without long lasting means of production, no great difficulty arises. In that case "roundabout way" is identical with "period of production" or "number of stages passed" (when each stage demands that a constant period be run through). The latter is, indeed, a useful notion for dynamic problems also. But, in reality, roundabout way depends in a high degree on the life time of machines used; the average time distance between productive act and consumption of the result, i.e., roundabout way, is greatly influenced by that life time. But, the specific notions, (a) life time, (b) production period for machinery, and (c) period necessary for the production of finished articles with the help of the machinery, are more apt for the description of dynamic processes than the single and general notion of roundabout way. The important questions of how production reacts on price and wage situations and how replacement activity behaves cannot be answered through the notion of roundabout way.

Two examples from Hayek's own line of argument may be given. He asks[44] what is the influence of an increase in consumers' income on the receipts of the producers of highest order means of production and replies that these producers will have increasing receipts only after the length of the roundabout way. This is unrealistic. As soon as the producers of consumers' goods have higher receipts, they invest more; and whether they replace worn-out machinery by new depends, in addition, on whether there is worn-out machinery at that moment. An element such as "confidence" in monetary policies might play a rôle, but I doubt that the length of roundabout way has much to do with it.

Again, Hayek seems to conclude from the lengthening of roundabout way in boom times that this entails a relative shortage of consumers'

[43] See the interesting remarks made by Hill, *Economic Journal* XLIII (1933), 599, on this question.

[44] *Preise und Produktion*, p. 58–68.

goods.[45] He seems to hold that the period of production is also length-
ened and that, therefore, in the time of "transition" from the shorter
to the longer period of production, there is a certain shortage. This is,
indeed, true in the simple case where no long living means of produc-
tion are used, but it is by no means necessary when such means are
used. It is easily shown that roundabout way may increase, whereas
period of production, i.e., time distance between beginning of raw
material production and end of finished goods production, decreases.
It is only necessary that the percentage of production cost relating to
amortization rise. For a number of industries the period of production
does, indeed, decrease during the boom period.[46]

We return to the unsolved questions rising from the fact that certain
relations are not expressly stated but are, nevertheless, necessary to
reach a closed system. This applies expecially to the way in which
money rates are formed. As to the demand and supply functions of
credit, Hayek is vague. This becomes a fundamental difficulty because
he attaches so much weight to interest rates ("long" or "short" rates?).
It is not clear whether he supposes them to depend on prices, or price
derivatives, etc., or on any or what combination of these variables.
It is not clear whether or not certain lags are supposed to exist here.

Thus, when one tries to find out what dynamic relations play the
the principal rôle in Hayek's system, and whether his system in all
circumstances leads to cycles—and, therefore, to an automatic return
of prosperity—there remain many difficulties, which is, after all,
Hayek's own view.[47]

11. Keynes probably does not pretend to give a closed theory. In his
Treatise on Money there are three parts which give very pertinent re-
marks on the business cycle problem. First, there is the part in which
the general exchange equations are discussed. So much has been writ-
ten on these equations[48] that I confine myself to one very general
remark. In reading this part of Keynes' work, one realizes clearly that
business cycle problems have not so much to do with the movements
of *prices* and *quantities* considered as separate variables but rather
with the fluctuation of *values* (prices multiplied by quantities).

The second part in Keynes which deserves special attention is Chap-
ter 27, dealing with the notion of "working capital." This seems very
important to business cycle theory and admits of mathematical treat-
ment. It is intimately related to the notions (used by Frisch, Kalecki,

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[45] *Preise und Produktion*, p. 84.

[46] Cf. "De duur van eenige productieprocessen," *De Ned. Conj.*, August 1934,
p. 32.

[47] Cf. his contribution to "Der stand und die nächste Zukunft der Konjunk-
tur forschung," *Festschrift fur Arthur Spiethoff* (München, 1933).

[48] Cf. Hansen and Tout, ECONOMETRICA I (1933), p. 119.

and myself) of the time shape of production. Writing $f(x)$ for the time shape of production, i.e., for the amount of productive activity applied per unit of product and per unit of time at x time units after the starting of production (the "advancement function" in Frisch's terminology), then the accumulated amount of activity invested in the production of one product unit at the moment t, production having begun at time τ, will be

$$\int_\tau^t dx\, f(x).$$

When $a(\tau)$ units per time unit at time τ have been started, the amount invested in *these* units will be

$$a(\tau) \int_\tau^t f(x)dx.$$

Production processes at many different moments τ will have been started and all will contribute to the general total of activity invested, provided the products have not yet been finished. Assuming a fixed lag between the initiating and the finishing of the process, this general total amounts to

$$\int_{t-\theta}^t a(\tau)dt \int_\tau^t f(x)dx.$$

When $f(x)$ is expressed in value units instead of physical units, we have working capital. Assuming $f(x)$ to depend only on τ, for example, the double integral reduces to a simple one, and so on. It is easily seen that, as soon as working capital enters our equations, these will be dynamic.

The function of working capital in Keynes' system is not, however, quite clear. Probably it has influence on the price formation of money rates, but, like Hayek, Keynes does not work this out rigorously. The only thesis stated clearly in the chapters on working capital is that its increase is bound to a certain maximum depending on the nation's savings. This is a type of quantitative reasoning of a remarkable sort which enters many arguments on business cycles. It is used sometimes and, in a certain respect also by Keynes, to calculate the length of periods. For that purpose, it is not only assumed that its velocity is bound to a maximum but also that the uniform growing process, corresponding with that maximum velocity, is ended when a certain maximum possible amplitude is reached.[49] Further, it is supposed, in addition, that the growing process is immediately followed by its reverse, a declining process. And, in a similar way, the length of the period is calculated. It has the advantage—aside from the question

[49] Cf. what is said about these questions in 22.

whether it forms a suitable approximation to reality—that the mathematics used are primitive. But, when one tries to write the equations on which the system rests, these are less simple as they contain discontinuous terms.

The third part in Keynes' *Treatise* relating to business cycle mechanism is found in Chapter 20, "An Exercise in the Pure Theory of the Credit Cycle." This chapter gives a very exactly described "standard case" of a moving system of a very simple nature. It is assumed that the quantity of consumption goods, the production of which is started, jumps, at a given moment t, from one constant value to another. It is shown that, corresponding to it, wage income and working capital increase steadily until the production lag has elapsed and they remain constant at the height corresponding to the new production. At that very moment, t_2, the supply of finished consumption goods makes the jump from the old to the new constant value. The price of these goods rises steadily during the period $t_1 t_2$ and falls suddenly to the old level at the moment t_2.

The system described in this standard case is not closed. We are not told what factors regulate the quantity produced. On the other hand, more attention is given to the financing of the production extension.

The standard case is generalized in different ways; influence of other methods of wage setting and financing and the influence of some form of speculation is considered; the length of the price cycle and the fact that the system is not closed are, however, not changed. Thus, we must look on this chapter—as its author does—as really an exercise. Some of its features will be met in the schemes to be discussed later.

12. *The Mathematical Theories.*—I now turn to a discussion of some business cycle theories that form macrodynamic closed systems and are formulated mathematically. The first publication of this sort was a paper in Polish by Kalecki in *Proba Teorij Konjunktury*, Warszawa, 1933.[50] A few months later appeared Frisch's "Propagation Problems and Impulse Problems." Both theories were presented at the Leyden meeting of the Econometric Society in 1933. Kalecki draws our attention chiefly to the following concepts.

Let $I(t)$ be the physical volume of investment orders at moment t. Then, assuming *a constant period of production* for investment goods, deliveries of newly produced investment goods, $L(t)$, will be equal to:

$$L(t) = I(t - \theta). \tag{12.1}$$

Assuming, further, a *constant volume of replacement* U, the *increase* K' of the total volume of capital goods present (the volume itself being K) will be:

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[50] For an English version of Kalecki's theory, see "A Macrodynamic Theory of Business Cycles," in this issue of ECONOMETRICA, pages 327–344.

$$K'(t) = L(t) - U. \tag{12.2}$$

On the other hand, we get, starting again from $I(t)$, an expression for the total volume of unfilled orders for investment goods (supposing each order is immediately put in hand) or of the total work in execution at t:

$$W(t) = \int_{t-\theta}^{t} I(\tau)d\tau. \tag{12.3}$$

The productive activity in investment goods industries will be proportional to this expression: $A = W/\theta$.

At this point, the most important assumption of Kalecki's system comes in. It is supposed that the volume of investment orders, $I(t)$, is determined by an equation of the type

$$I(t) = C' + mA(t) - nK(t), \tag{12.4}$$

where C', m, and n, are constants; C' is here written with a prime because it is not exactly the same as Kalecki's C. The justification of equation (12.4) is given by Kalecki by introducing it in the form:

$$\frac{I(t)}{K(t)} = m' \frac{C_0' + W(t)}{K(t)} - n, \tag{12.5}$$

indicating that the relative investment (relative to capital already present) is governed by the expression on the right side representing a linear function of relative profits.

It is easily seen that equations (12.1) to (12.4) lead to a mixed difference and differential equation. Kalecki shows that one of the components which this system may show—and the one which economically is the most important—is a periodic movement with a period which, assuming realistic values for the constants, turns out to be 10 years.[51]

The reader will observe that there are two dynamic features in Kalecki's system, viz., the production lag and the appearance of K—which has an integral character—in equation (12.4). These features have been mentioned in 4; they are essential to this theory insofar as the omission of one of them would lead to a non-oscillatory system. My impression is that the supposed influence of K is rather questionable.[52] A happy feature of Kalecki's system is the fact that he places capital goods production in the center. A very remarkable feature is that the

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[51] A fuller discussion of these numerical results is given by Kalecki, *loc. cit.*, and by Frisch and Holme, ECONOMETRICA III (1935), 225–239.

[52] This is illustrated by the way in which revival generates in this scheme. Although profits are still decreasing, it is by the decrease in capital that percentage profits increase and in Kalecki's argument investment is stimulated again. Is this sufficiently realistic?

very small number of variables included is sufficient to get a "closed system." The fact that other very much discussed concepts do not appear explicitly in it does not mean that they would not show any business cycle. For money rates, Kalecki states *expressis verbis* that they are assumed to be parallel to $W(t)$. The same thing might be assumed about prices which, remarkably enough, do not appear at all in his theory. Although general economic theory is accustomed to give the central place to prices and although I myself, in an attempt to give business cycle schemes, did so too, I do not consider this feature of Kalecki's system a great disadvantage. In many cases, statistical investigation leads to the suggestion that the rôle of prices is not so important as we have become accustomed to think. It is, in fact, quite possible that investment activity is predominant, a fact recognized and emphasized by many modern authors.

Like many other mathematical schemes, Kalecki's system will hold good even when quite different assumptions are taken as a base. Thus, the term with K in equation (12.4) might be interpreted by pointing to the influence of productivity on profits (cf. 4).

A striking difference from one of the schemes to be discussed further on, in 18, consists in the assumption of constancy of replacement. The schemes later to be mentioned assume just the opposite, i.e., that replacement demand after the life time of equipment has elapsed is the principal factor in the new revival. It is interesting to note that these schemes do not need the two dynamic features that are essential in Kalecki's system.

The Kalecki solution is, in one sense, of restricted value; he chooses such a solution as has zero dampening. As Frisch points out,[53] the solutions of the Kalecki equations, particularly their dampening characteristics, are very sensitive to variations in the data; it is an interesting question whether this is in accordance with facts. On the other hand, the period is not so sensitive, and this is an advantage in comparison with my own schemes.

To conclude, the structure of Kalecki's system entails certain difficulties of great practical importance. It is not clear, for example, how a price stabilization should be represented by it. What constants will be changed by price stabilization, or what additional terms will appear in the equations? How should other disturbing factors, such as changing crops, react on them?

The reader will have remarked, nevertheless, that there are important features in the theory and that the exact form in which it is presented creates the possibility of a clear and fruitful discussion.

13. A second case of a really closed system of equations representing

[53] *Op. cit.* (note 51).

business cycle mechanism was given by Frisch.[54] Its important feature is that it makes a distinction between the mechanism and the external shocks operating on that mechanism and shows, by a very interesting example, what shapes of cycles appear when such shocks are assumed to occur. For that reason, the special mechanism assumed by Frisch may have been intended to have merely an illustrative character. Its economic foundation is not clear in every point. Three variables are distinguished:

$x(t)$, volume of consumption,

$y(t)$, production of capital goods started, the same variable as Kalecki's $I(t)$, and

$z(t)$, called by Frisch "carry-on-activity", equivalent to Kalecki's variable $A(t)$.

The following equations are assumed to exist between x, y, and z:

$$z(t) = \frac{1}{\epsilon} \int_{t-\epsilon}^{t} y(\tau)d\tau \qquad (13.1)$$

(ϵ being the construction period of capital goods),

$$y(t) = mx(t) + \mu\dot{x}(t), \text{ and} \qquad (13.2)$$

$$\dot{x}(t) = c - \lambda(rx + sz). \qquad (13.3)$$

The significance of the first of these equations is clear. That of the second is based on the hypotheses that replacement of capital goods is proportional to consumption $x(t)$ and new investment is proportional to $\dot{x}(t)$. I discussed the relative validity of these hypotheses in 4. Sufficient statistical evidence on (13.2) has not yet been given.[55] The third equation is based on the notion of "encaisse désirée," introduced by Walras, and it sets forth that as the need for that encaisse désirée $\lambda(rx+sz)$, depending on consumption and production, increases, the rate of growth of consumption $\dot{x}(t)$ diminishes. This may be true. But it is desirable that these relations be analyzed economically and investigated statistically before they are made a basis for a theory, especially when the number of competing possibilities is as large as here. It might be that the "brake" included in the mechanism of equation (13.3) works in a different way.

Again, Frisch chooses his constant ϵ, the construction period of capital goods, equal to 3 years. I think this is too long, even though one considers—as Frisch does in this connection—the planning and erection of new plants and new enterprises. No doubt the construction

[54] "Propagation Problems and Impulse Problems in Dynamic Economics," *Economic Essays in Honour of Gustav Cassel* (1933), p. 171.

[55] Cf. what has been said in 4 C.

of some parts of equipment needs several years, but there are numerous cases where a certain extension of consumption can be obtained by much shorter investment processes. The discussion of this point is difficult, however, because the relation is not easily verified. One of the direct consequences of the assumption of this lag is, for example, that production starting, $y(t)$, and carry-on-activity, $z(t)$, move almost in opposite directions. I think this does not correspond with reality. Moreover, the length of the periods found is highly dependent on the value of ϵ.

Nevertheless, Frisch's method is important and so is his result. The mechanism appears to lead to movements with periods of $3\frac{1}{2}$ and $8\frac{1}{2}$ years and the influence of m, μ, c, λ, r, and s, on these periods is but small.

14. A remarkable scheme has been presented by Roos,[56] who considers rather in detail the entrepreneur's fixing of prices and production. Assuming a demand function

$$y = \gamma(ap(t) + b + h\dot{p}(t)) = \gamma u, \qquad (14.1)$$

and a cost function

$$Q = Au^2 + Bu + C + D\dot{u}^2 + F\dot{p}^2 + G\dot{p} + H\dot{u}, \qquad (14.2)$$

he calculates profits over a given time interval

$$= \int_{t_1}^{t_1} (\gamma pu - Au^2 - Bu - C - D\dot{u}^2 - F\dot{p}^2 - G\dot{p} - H\dot{u})E(t_1, t)dt \qquad (14.3)$$

which is maximized under the condition that (14.1) is satisfied; u being quantity produced, p price, y quantity sold, $E(t_1, t)$ a "discount function," reducing money amounts at time t to time t_1 and being equal to $1 - \int_{t_1}^{t} \delta(r)dr$, where $\delta(r)$ is the interest rate.

All other symbols represent, for the moment, constants.

The problem thus posed is a variable end point variation problem. Its solutions must satisfy a fourth order differential equation. They may have different forms, as exponential and periodical components can be combined in different ways.

Aside from many details and some questions of principle treated in a very elegant way by Roos, his view of the problem of economic oscillations is that every change in constants causes jumps in other individual solutions. In that way oscillatory and exponential, damped and anti-damped movements follow one another.

It cannot be denied that there are some very suggestive elements in

[56] "A Mathematical Theory of Price and Production Fluctuations and Economic Crises," *Journal of Political Economy*, Oct. 1930, pp. 501–502.

Roos' theory. Above all, the idea of the successive pursuance of curves of different shape is of importance and its analogy with Frisch's "shock" theory is evident.

As to the mechanism itself, 1 have several objections. It seems a little unbalanced in its economic construction. The questions as to how the changes in production affect incomes and, therefore, demand, and how they influence money rates, are not considered.[57] On the other hand, the assumed cost function is very complicated, much more complicated than is necessary, as a first approximation, to describe facts. This applies particularly to the constants D and F. They play an important rôle in Roos' equations; when $D = F = 0$ they generate to much simpler ones. When, in addition, also $A = 0$, no oscillations would occur at all. Furthermore, in Roos' theory no estimation of the length of the periods is undertaken. This would meet with several problems, of course, as a statistical determination of the constants introduced would be very difficult.

In some respects, Vinci has also presented a mathematical theory of business cycles, in the sense that he has given a system of eight equations.[58] Vinci has not yet published anything on the *solutions* but, nevertheless, I should like to make a few remarks on his attempt. The interesting feature in it, which is, no doubt, an advantage in comparison with the simpler schemes, is the larger number of equations. To the assumptions on which his special equations are based, I have serious objections. He supposes, for example, that the physical volume of production is determined by the velocity with which profits change and the velocity with which income spent changes (Eq. I). There is very little evidence that these velocities are the chief factors. The same might be said of his equations II, V, VI, VII, and VIII. It should be added that Vinci himself sees the necessity of empirical investigations in this respect.

15. *Other mathematical schemes. Short lag schemes.*—The schemes of Kalecki, Frisch, Vinci, and myself,[59] are examples of the wide variety of possibilities of explaining economic oscillations. The various schemes put different emphases on the various factors. In my own schemes, prices occupy a central place. It is convenient to divide these schemes into *short lag schemes* and *long lag schemes.* "Short lags" are short in comparison to the period of the cycles, whereas the "longer" ones are of the same order of magnitude. The first sub-group admits of a special

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[57] At this time Dr. Tinbergen had not seen Roos' *Dynamic Economics*, Bloomington, 1934. ASSISTANT EDITOR.

[58] "Significant Developments in Business Cycle Theory," ECONOMETRICA II (1934), pp, 136–139.

[59] *Zeitschrift für Nationalökonomie* v (1934), 289; Wagemann-Festschrift.

method of treatment that will presently be discussed. For the reason already mentioned, all schemes are very much simplified. They are macrodynamic. No difference is made between separate categories of goods (except between such categories as investment goods and consumption goods, or, in some cases, between raw materials and finished goods), nor between separate categories of factors of production (in many respects labor and capital behave quite similarly). In later sections of this survey some of the resulting limitations will be removed. It may be emphasized, however, as is pointed out also in the papers quoted, that the formulas obtained often have a much wider validity than the simplified scheme for which they are deduced.

The mathematical method of analysis is simplified by the introduction of the notion of "equilibrium values" for each variable. By equilibrium values, the constant values (contrary to the variable values or functions of time) that satisfy the equations are meant. Such equilibrium values will be indicated by capital characters and the variable values will be measured as deviations from these equilibrium values. These deviations are indicated by the corresponding small characters.

Pure lag schemes.—The simplest cases considered, which are useful also as a methodological starting point, are those in which the only dynamic feature included is a production lag. This entails two consequences: the number of finished products shows this lag as compared with the number of products started and the number of production factor units used (chiefly labor) shows, as a first approximation, half this lag. We introduce the following symbols:

Price of finished consumers' goods, $P + p(t)$,

Number of products started (consumers' goods), $Z + z(t)$,

Number of products sold (consumers' goods), $Y + y(t)$,

Income spent by consumers, $X + x(t)$,

Increase of stocks of products, $V + v(t)$.

It will easily be seen that $Z = Y$, whereas $V = 0$.

Furthermore, suppose the following relations to exist between these variables:[60]

$$x(t) = \frac{2k}{a} z(t - 1), \tag{15.1}$$

$$\epsilon v(t) = z(t - 2) - y(t), \qquad (\epsilon = 1 \text{ or } 0) \tag{15.2}$$

$$\epsilon(z(t - 2\eta) - ap(t)) = 0, \qquad (\epsilon = 1 \text{ or } 0) \tag{15.3}$$

$$z(t) = \epsilon'ap(t) + (1 - \epsilon')y(t), \qquad (0 \leq \epsilon' \leq 1) \tag{15.4}$$

$$(Y + y(t))(P + p(t)) = X + x(t). \tag{15.5}$$

[60] These relations are all assumed to be linear; a more general conception is given by Haldane, *Review of Economic Studies* (1934), p. 186.

The economic meaning of these equations is the following. Eq. (15.1) expresses a hypothesis regarding the oscillations of income spent, viz., that they are proportional[61] to the oscillations of productive activity. This holds exactly, for example, when all factors of production have constant earning rates and when entrepreneurs are also supposed to spend in proportion to activity. This may seem an unrealistic hypothesis. More detailed considerations will show, however, that numerous cases of realistic value can be brought under this formulation (cf. 8). As an example, it may be remarked that our formulas hold when the amounts spent by entrepreneurs (e.g., farmers) show other movements, provided that these are synchronous with the value of products sold. In that case, equal terms are to be added to the left and right sides of equation (15.5) and these terms neutralize each other (cf. what is said below on equation (15.5)).

Equation (15.2) expresses, for $\epsilon = 1$, a purely technical relation between production finished, $z(t-2)$, sales, $y(t)$, and stock increase, $v(t)$. For $\epsilon = 0$, it indicates that no stock variation occurs.

Equation (15.3) may be called the price setting equation. It expresses the fact, for $\epsilon = 1$, that, when production is high, prices are fixed at a high level, and when production is low, prices are also low. This relation may be considered as an expression for the result of price calculations by the producer. According to different choices of η, price setting shows more or less of a lag behind planning. For $\epsilon = 0$, this equation is abolished, as price setting by the producer is impossible in the case that all products have to be sold. Market relations then determine prices. The introduction of ϵ, therefore, simply means the introduction of alternative forms of selling policy.

Equation (15.4) may be called a production planning equation. It gives the relation between market condition, represented by price $p(t)$, and sales, $y(t)$, on one side, and new production started, $z(t)$, on the other, which is an expression for the considerations leading to the fixing of $z(t)$. It is assumed that both "past performance" and price situation may act on $z(t)$, but, by varying ϵ', the one or the other factor may be given more stress. As limiting cases, we have $\epsilon' = 1$ where production is entirely determined by prices, and $\epsilon' = 0$ where it is entirely determined by "past performance."

Equation (15.5) may be called the market equation. It expresses the fact that the amount spent by consumers equals the product of prices and quantities of consumers' goods sold. As the reader will already have seen, equations (15.1) to (15.5) represent different cases, depending on the special choice made for ϵ and ϵ' in the first place, and

[61] The proportionality factor is written $2k/a$ to accord with the paper mentioned.

for the other constants. Values for ϵ and ϵ' are not quite independent, as $\epsilon' = 1$ must necessarily entail $\epsilon = 0$ or $\eta = 0$. Indeed, $\epsilon' = 1$ means that $z(t) = ap(t)$ and this would be contrary to (15.3) unless $\epsilon = 0$ or $\eta = 0$. A certain choice for ϵ does not, however, restrict possibilities for ϵ'.

In terms of the "literary" business cycle theories, these schemes may be characterized as follows. In the first place, they show the case of a general over-production and over-investment. Second, in these schemes the variable p is proportional to the deviation of the natural rate of interest (in the Wicksell-Mises-Hayek sense) from its own long run equilibrium value and also from the money rate of interest. Third, in one of the more complicated versions (viz., after the introduction of integral terms below) the notion of scarcity of capital (Spiethoff) is introduced, although in a very simple way. In this respect, there are many opportunities of improvement of the schemes. This question of the introduction of the influence of capital scarcity is a suitable example of the vagueness of many "literary" theorists as to the description of relations introduced. Many mathematical forms may compete here: By what variables is the demand and the supply of credit determined? Are the changing velocities of these variables an important factor to be included? Nothing explicit is to be found in the literature; nevertheless, it makes a big difference which one of these demand and supply functions is used as basis. Some, for example, yield no cyclical movements at all. For others, quite different phase differences between some of the variables will occur, and so forth.

We proceed to the treatment of two special cases of the scheme given by (15.1) to (15.5).

A. $\epsilon = 0$, $\epsilon' \neq 1$.

Equations (15.1) to (15.4) then become: $x(t) = 2k/a \, z(t-1)$; $z(t-2) = y(t)$; $z(t) = \epsilon' ap(t) + (1 - \epsilon')y(t)$, and, substituting this in Eq. (15.5), we get for small movements:

$$z(t - 2) + \frac{z(t)}{\epsilon' a} - \frac{1 - \epsilon'}{\epsilon' a} z(t - 2) = \frac{2k}{a} z(t - 1), \text{ or}$$

$$z(t) - 2k\epsilon' z(t - 1) + (\epsilon' a + \epsilon' - 1)z(t - 2) = 0. \quad (15.6)$$

The solution is found[62] by putting $z(t) = C\xi^t$, which leads to the following equation for ξ:

$$\xi^2 - 2k\epsilon'\xi + (\epsilon' a + \epsilon' - 1) = 0,$$

$$\xi = k\epsilon' \pm \sqrt{k^2\epsilon'^2 - \epsilon' a - \epsilon' + 1}.$$

[62] Cf. Frisch and Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," ECONOMETRICA, *loc. cit.*, for this and the following problems.

It follows that the movements are periodical when

$$k^2 \epsilon'^2 + 1 < \epsilon'(a + 1),$$

and exponential when

$$k^2 \xi'^2 + 1 \geqq \epsilon'(a + 1).$$

In addition, we have $|\xi|^2 = \epsilon'(a+1) - 1$.

The special case $\epsilon' = 1$ is identical with that treated in my *Zeitschrift* paper. A fuller discussion of the solutions will be given, together with that of the other cases, in 16.

B. $\epsilon = 1, \epsilon' \neq 1$.

Equations (15.1) to (15.4) now become:

$$x(t) = \frac{2k}{a} z(t - 1); \ v(t) = z(t - 2) - y(t); \ z(t - 2\eta) = ap(t);$$

$$z(t) = \epsilon' ap(t) + (1 - \epsilon')y(t).$$

Of these, the second is to be considered as the definition of $v(t)$ only. Substitution in (15.5) gives, after some rearrangement,

$$z(t) - \frac{2k}{a} z(t - 1) + \left(\frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'} \right) z(t - 2\eta) = 0. \qquad (15.7)$$

For $\eta = \frac{1}{2}$, no oscillatory movements (except, in some cases, with period 2) are possible. For $\eta = 1$, we get for ξ, introduced by $z(t) = C\xi^t$,

$$\xi^2 - \frac{2k}{a} \xi + \frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'} = 0,$$

$$\xi = \frac{k}{a} \pm \sqrt{\frac{k^2}{a^2} - \frac{1}{a} + \frac{\epsilon'}{1 - \epsilon'}}.$$

The "wave condition" now becomes

$$\frac{k^2}{a^2} - \frac{1}{a} + \frac{\epsilon'}{1 - \epsilon'} < 0, \quad \text{whereas},$$

$$|\xi|^2 = \frac{1}{a} - \frac{\epsilon'}{1 - \epsilon'}.$$

Introduction of integral terms.—The schemes just given are admittedly much simplified. We shall now show different possibilities of making closer approximations to reality. A first possibility is no longer to suppose remunerations of the factors of production to be constant. As is shown in more detail in the paper cited above, a good approxima-

tion to reality consists in supposing that wage rates show year-to-year changes proportional to employment deviations from trend (cf. also section II, 5 of this survey). Using profit margin as the central variable, this second approximation means an introduction of integral terms $\int_0^t d\tau\, p(\tau)$, where p is the profit margin.

The same occurs when productivity changes are taken into account or, as in Kalecki's case, total capital accumulated since some initial date.

Needless to say, when an integral term is added, oscillations are also possible in cases where, for example, only two terms of the type of equation (15.6) or (15.7) are present.

Introduction of derivatives.—Another type of approximation leads to the introduction of derivatives. Examples of great importance are different sorts of speculative influences. Apart from prices or profit margins, the change in prices or margins may have influence on productive activity— as is assumed also, though not always on exactly the same grounds, by Amoroso,[63] Fisher,[64] Roos,[65] Evans,[66] and Vinci[67] (cf. 16). The interpretation of such relations is given in 4.

We shall treat, also, a set of examples of these cases. Amending Case A (for $\epsilon' = 1$, i.e., the "pure lag case" treated in my *Zeitschrift* article) we get Case

C. $z(t) = ap(t) + ab\ \dot{p}(t)$; $y(t) = z(t-2)$; $x(t) = 2kp(t-1) + bk\ \dot{p}(t) - bk\ \dot{p}(t-2)$, leading to the following final equation for $p(t)$, which we take now as the central variable.

$$(1 - bk)p(t) - 2kp(t - 1) + (a + bk)p(t- 2) + ab\dot{p}(t - 2) = 0. \quad (15.8)$$

In 16 a further discussion of this case will be given.

Case D. We may generalize Case C by supposing the supply of consumables to be, in the units already introduced,

$$1 + ap(t + n) + a'\dot{p}(t + n), \quad (15.9)$$

and the amount spent for consumables,

$$1 + 2kp(t + m) + 2k'\dot{p}(t + m). \quad (15.10)$$

The meaning of these formulas is clear, when n is negative, $m = \frac{1}{2}n$, and $k'/K = a'/a$, as they then turn into those of Case C (with a high

[63] "Ciò che è chiaro e ciò che è oscuro nelle fluttuazioni dei prezzi," *Atti dell' Istituto Nazionale delle Assicurazioni* IV (1932).

[64] *Theory of Interest*, 1930.

[65] Though on somewhat other grounds; cf. "A Mathematical Theory of Competition," *Amer. J. of Math.*, XLVII (1925), and many other publications. See also note 4.

[66] Cf. *A Math. Introduction to Economics*.

[67] ECONOMETRICA II (1934), p. 136.

degree of approximation). When these conditions are not fulfilled, however, it is not difficult to give them a suitable interpretation.

For example, n may also be positive. This means that production no longer is fixed by the price situation but, conversely, as we already discussed, that prices are determined by production, i.e., (15.9) is then analogous to (15.3). The term $a'\dot{p}(t+n)$ may then be attributed to the circumstance that price setting is influenced by historical cost prices.

When, in this case, $m > n$ and $k'/k = a'/a$, our previous expression for the amount spent still holds good. This amount is then determined by the productive activity corresponding with the supply of consumables, activity in capital goods industries being assumed parallel to that activity.

It is, however, also possible that $m \leqq n$. This may be the case when there are retarded elements in the amount spent, such as the lagging of salaries behind activity,[68] or even some forms of saving.

It is equally possible that $k/k' = a/a'$ no longer holds. This is possible when, in the amount spent, certain elements appear that are not parallel to activity. When these elements are parallel to prices (as wage rates may be), k will be larger than would correspond with the proportionality just mentioned. When they are parallel to price changes (as speculation gains may be), k' will be larger.

The possibilities mentioned show that, until all questions raised are answered by detailed factual research, there is a very large number of possible schemes. This number may be further enlargened by introducing the peculiarities corresponding with $\epsilon' \neq 1$ in Case A. Our considerations are, therefore, only to be looked upon as *examples*.

Expressions (15.9) and (15.10) lead us to the equation (neglecting second order terms),

$$ap(t+n) + a'\dot{p}(t+n) + p(t) - 2kp(t+m) - 2k'\dot{p}(t+m) = 0. \qquad (15.11)$$

Mixed form of equations

By introducing integral, derivative, and lag terms, still more general types of equations, or systems of equations, are obtained. Those will be reached when, as wages and productivity change, speculative influences and the fact of different lags for different industries or enterprises are considered. As far as one equation with only one variable finally results from elimination, this equation will have the form:

$$\sum_1^n a_i\,p(t-t_i) + \sum_1^n b_i p(t-t_i) + \sum_1^n c_i \int_0^{t_i} p(\tau)d\tau = 0. \qquad (15.12)$$

[68] In this case it is supposed that wages show a lag behind activity and not, as was done in other cases, that they have "integral character." Cf. 5.

16. *"Wave Condition" and "Long Wave Conditions."*—As has been shown already in some of the cases treated, the solution of the equations or systems of equations may have different time shapes. The simplest cases often show solutions that are exponentials for some value range of the constants and oscillations (damped or even anti-damped) for other ranges. The conditions determining those latter ranges may be called wave conditions. For Cases A and B, the wave condition has been given.

To explain real business cycles by one of the schemes discussed here, not only must the wave condition be satisfied by the constants but the waves must, in addition, be fairly long in comparison to the lags introduced, i.e., to the time unit. This enables us to prescribe more rigorous conditions for the constants, which we may call "long wave conditions." By this term we mean not only the condition for a long wave but also for a wave not differing too much from an undamped one. Waves which are practically damped down long before one cycle has been described should be excluded.

The first approximation to the long wave conditions is found by putting $r = |\xi| = 1$ and $\varphi = \arg \xi = 0$ (see 15, sub A.). In this case, however, the mistake must be avoided of applying this method to cases where no oscillation at all is possible. The position $|\xi| = 1$, $\varphi = 0$, must be a limit of values where φ really differs from zero, and not one of values where only r varies. To make sure, the wave condition must be ascertained first and then it is clear that one of the long wave conditions is nothing other than the wave condition with the sign reduced to an equality sign.

So we find "long wave conditions (first approximation)" for

Case A: $\quad a = \dfrac{2}{\epsilon'} - 1, \qquad\qquad k = \dfrac{1}{\epsilon'};$

Case B: $\quad a = k = 1 - \epsilon';$

Case C: $\quad a + b = 1, \qquad\qquad 2k = a + 1;$

Case D: $\quad a + 1 - 2k = 0, \quad na + a' - 2km - 2k' = 0.$

These conditions will be a guide in a statistical test of the different schemes as to their accord with reality. (cf. 17). It may be remarked beforehand that the application of this method to the general equation (15.12) leads to the equation:

$$\sum_i^n c_i = 0$$

as one of the long wave conditions. This means that the sum of the coefficients of the integral terms must be zero; when only one integral

term is present, its coefficient should be zero. As we have to do with a
first approximation only, it need not be exactly zero but it should be
small; schemes of the form (15.12) only then lead to long, not too
much damped waves when integral terms are of small importance.

This has a remarkable consequence for those schemes in which wages
are supposed to be variable and of the special form discussed in 5 (2),
i.e., that they move parallel to the integral of profit margins or of
activity. When such schemes have, in addition, no other integral terms
of opposite sign and no other dynamic elements such as are included
in the general equation (15.12), it will be impossible to explain real
business cycles with their help. To state it in another way, the influence
of wage changes as they have occurred in reality (I make an exception
for the extraordinary wage changes caused by the codes in the U.S.A.
in 1933) cannot be one of the chief factors in business cycles.

It is possible to give a *second approximation* by assuming $r = 1 + \delta$,
where δ is a small quantity and φ is also a small value. It is in accord-
ance with reality to suppose more exactly that δ is of the same order
of magnitude as φ^2 and φ^3 is negligible. As an example, Case C has been
treated in this way. The result is that the long wave conditions (2nd
appr.) are (writing—6ψ for φ^2):

$$(1 - bk)(1 + 2\delta + 12\psi) - 2k(1 + \delta + 3\psi) + a + bk + ab\delta = 0, \text{ and}$$
$$(1 - bk)(2 + 4\delta + 8\psi) - 2k(1 + \delta + \psi) + ab = 0.$$

It is possible to solve these equations for a and k:

$$a = \frac{b(1 + 2\delta + 4\psi) - (1 + 3\delta + \psi)}{6\psi b^2 - b(2\delta + \psi) - (1 + \delta + \psi)} \text{ and}$$

$$k = \frac{2 - b}{2} + \frac{b - 1}{2} (\delta(2b - 1) - \psi(6b^2 - 7b + 6)),$$

from which the following table is calculated, showing values for b, a,
and k, that lead to given values for r and φ.

17. *Statistical verification of "long wave conditions."* To find out
whether these schemes can explain real business cycles and which of
them most resembles reality, a statistical determination of the con-
stants used in our formulas becomes necessary.

There are two somewhat different ways to measure these constants,
which may be called the *structural* and the *historical* method. The dif-
ference may be illustrated by discussing the measurement of a. This
constant measures the quotient of corresponding percentage changes
in prices and production along the supply curve, other things affecting
supply being assumed constant. The cause of this correspondence is

TABLE I(A)	TABLE I(B)
VALUES FOR a, b, AND k, CORRESPONDING WITH $r = 0.9$ AND DIFFERENT PERIODS	VALUES FOR a, b, AND k CORRESPONDING WITH A PERIOD OF 26 AND DIFFERENT r's.

		Periods[a]			Value of r			
		21	26	36	0.90	0.95	1.05	1.10
$b =$								
0.10	$a =$	0.88	0.89	0.90	0.89	0.91	0.94	0.96
	$k =$	0.71	0.71	0.71	0.71	0.81	0.98	1.06
0.20	$a =$	0.85	0.86	0.87	0.86	0.87	0.89	0.90
	$k =$	0.64	0.63	0.63	0.63	0.72	0.87	0.94
0.30	$a =$	0.81	0.82	0.83	0.82	0.82	0.84	0.85
	$k =$	0.56	0.56	0.55	0.56	0.62	0.77	0.82
0.40	$a =$	0.78	0.78	0.79	0.78	0.78	0.79	0.79
	$k =$	0.48	0.47	0.47	0.47	0.55	0.66	0.70
0.50	$a =$	0.74	0.74	0.74	0.74	0.74	0.74	0.74
	$k =$	0.39	0.39	0.38	0.39	0.46	0.55	0.59
0.75	$a =$	0.62	0.63	0.63	0.63	0.62	0.62	0.61
	$k =$	0.17	0.16	0.15	0.16	0.22	0.30	0.33

[a] The time unit is chosen equal to half the production lag.

to be seen chiefly in the fact of unequal cost of production in different enterprises in the same industry. When prices increase, a larger part of enterprise becomes profitable. The velocity with which this part grows, or the increase in this part for a unit change in prices, measures a. This a might, therefore, be determined from a cost distribution function, in any case from a timeless structural datum. It might, however, also be determined in the usual statistical way of determining supply curves from historical data on prices and production figures considered to correspond to these prices. In general, since data at our disposal do not permit following the structural method, the historical method has been chosen.

There are many difficulties to be faced which, to a large extent, are caused by two circumstances, the lack of adequate data and the great simplification of reality adopted by the schemes.[69] In the cases we are

[69] No attention is given, for example, to the special character of the building trades, which largely produce consumers' goods but goods that are not in general sold to the consumers and do not appear in our equation. On the other hand, in some cases, no attention is paid to the fact that a considerable part of consumers' income is spent on housing rent, forming a rather rigid amount. This circumstance can be taken account of by subtracting a constant figure from the

now interested in, the customary way is to reach exact statements: the flight into detailed study, is now, in principle, closed. I am, therefore, convinced of the highly approximate character of the following attempts.

The first verification attempt relates to the pure lag schemes A and B. It attempts to determine constants a and $2k$ of these schemes. In either of these cases, a manifests itself as the quotient of two amplitudes, that of the sales volume of consumers' goods and that of prices. This will be easily seen when it is remembered that the lags appearing in the equations are but small, so small that, in connection with the irregular components in each series, they cannot be estimated with any accuracy. The amplitudes should be measured in units equal or proportional to the equilibrium value of either variable. This equilibrium value has been assumed constant in our schemes but it is easily seen that no changes occur in these schemes by assuming that value itself variable. To reach a larger agreement with reality, we do so and suppose trend values to be equilibrium values.[70] In some cases, irregular movements have first been eliminated by using three year moving averages. Trends are obtained by straight lines. Values of a and $2k$ have been obtained by simply dividing the average absolute value of percentage deviations from trend for the series mentioned in the headings of the table by the corresponding absolute value for the price series. When there are different series that can be estimated to approximate the variable in question, alternative calculations have been made. It should be borne in mind, finally, that prices must be prices of finished products paid by consumers, so retail prices or even cost of living indexes have been used. In a similar way, $2k$ is the amplitude quotient for income spent and prices. The results are given in Table II.

The figures given are very rough. In some cases, standard deviations have been calculated, in other cases the degree of uncertainty is illustrated by the influence of the omission of one or more years or by the deviations between competing series for the same economic category. Standard deviations are multiplied by 3.

Comparing these results with the long wave conditions for Case A, we can state that, as these conditions require both for a and for k values > 1, this case cannot explain business cycles in England and the

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amount spent by consumers; the influence on our constant k will be that this constant will become larger in about the same proportion. In the same way, other corrections are possible, but it seems wiser to postpone these corrections until the whole subject is treated in a more detailed manner.

[70] The problem that lies behind this hypothesis is fully recognized, but it is considered of secondary importance in the question now discussed.

TABLE II

VALUES FOR a AND k IN SCHEMES A AND B.

The left half of each table relates to a, the right half to k. In the column heading, the series used for the quantity of consumers' goods sold (in the case of a) or the series used for income spent (in the case of k) is indicated; in the rows, the series used for prices is mentioned. The period covered is indicated in the column headings.

ENGLAND

	Production of consumers' goods calculated from London & Cambridge Economic Service Bulletin 1921–1931	Production of consumers' goods (Vierteljahrsh. z. Konj.) 1921–1930	National Income (cf. chart) 1921–1930	1921–1931	Actual income of tax payers 1921–1930
Cost of Living	0.6	0.8	0.3	0.5	0.4

GERMANY, 1924–32

	Retail sales: cost of living (without rent)	Production of consumers' goods (old series, I.f.K.)	Production of consumers' goods (new series, I.f.K.)	National Income (without Reparation Payments)	Labor Income	Income spent for Free[a] Consumption
Cost of living, food only	0.9	1.5	0.9	1.0	1.2	0.8
Cost of living, excl. rent	1.0	1.7	1.0	1.1	1.3	1.0
Cost of living, total	1.0	1.7	1.0	1.1	1.3	0.9

[a] Cf. *Vierteljahrshefte zur Konjunkturforschung*, 1934, (8 Jg, H. 3), p. 161.

UNITED STATES

	Production of consumers' goods, Wagemann index, 1920–1930	Production of consumers' goods League of Nations index, 1921–1932	Production of foods 1919–1933	Department Store Sales: Retail prices (food)	Pay rolls		National Income minus estimated savings (Walsh)		
					1919–1933	1920–1932	1920–1930	1920–1932	1920–1930
Retail prices, food	0.4	1.1±0.8	0.4	0.5	0.9	0.8 ± 0.7	0.6	0.5	0.4

United States, but it can, however, explain them in the case of post-war Germany. For that country, values for a and k are near those required in the special case where $\epsilon' = 1$ (the *Zeitschrift* case).

For Case B, on the contrary, a and k must have values below 1, and this is in accordance with English and American figures as well as for a "world" figure obtained by averaging English, American, and German figures.

In general, it may be stated that the order of magnitude of the constants is the one necessary to our schemes but detailed research will be necessary to make final decisions.
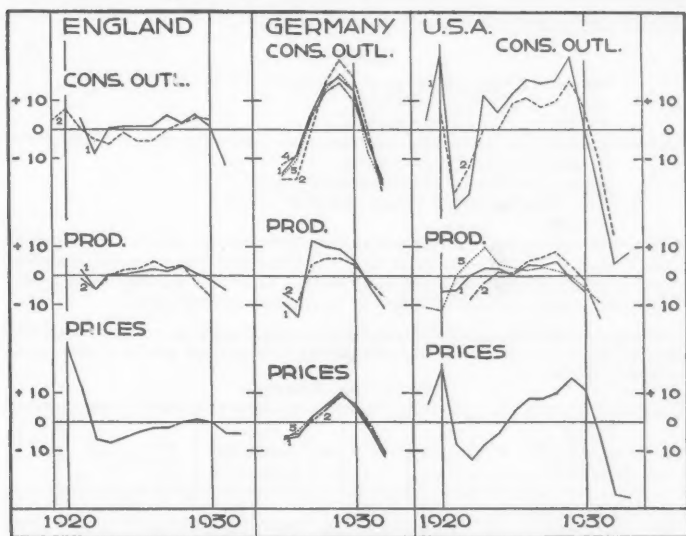
FIGURE 4.—Some data on consumers' outlay, supply of consumers' goods, and retail prices, used in the statistical verification of schemes A and B.

Germany.—"Consumers' Outlay": (1) National Income minus Reparations; (2) Labour Income; (3) National Income; (4) Estimated Consumers' Outlay.

"Production": (1) Production of Consumers" Goods, old series (Konjunkturstat. Handbuch); (2) Production of Consumers' Goods, new series (Wochenbericht, 9–5–1934).

"Prices": (1) Cost of living: food only; (2) Cost of living without rent; (3) Cost of living, total.

England.—"Consumers' Outlay": (1) National Income (1920–23: *The Economist;* 1924–'31: Colin Clark); (2) "Actual Income" of Tax Payers.

"Production": (1) Production of Consumers' Goods, calc. by myself from London and Cambridge Econ. Service indexes; (2) Idem, calc. by Institut f. Konjunkturforschung, Sonderheft 31.

"Prices": Cost of Living.

United States.—"Consumers' Outlay": (1) Total Pay Roll; (2) Income minus Savings (Walsh, *Qu. J. of Econ.* XLVIII (1934), p. 546).

"Production": (1) Production of Consumers' Goods (*Inst. f. Konj.*, Sonderheft 31); (2) Production of Consumers' Goods (League of Nations, *Monthly Bull.* 1934, p. 338); (3) Production of Foods.

Although a rough approximation to reality may be obtained with Scheme B, several objections rise on closer examination of the figures. The correlation between the three variables considered is not satisfactory, as shown in Figure 4. Again, phase differences in scheme B are not the same as in reality, as far as it is possible to tell at all.

This explains the second attempt, an attempt to verify the more complicated schemes C and D. The same remarks on the difficulties resulting from the inadequacy of data and the approximate character of the schemes must be repeated here. In addition, the figures to be given for a' and k' depend on the time unit chosen. I have used one quarter of a year. The figures given are the regression coefficients obtained by a simple least squares calculation.

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TABLE III

VALUE OF COEFFICIENTS a, a', $2k$, AND $2k'$, IN THE EQUATIONS:
</div>

$$u = ap(t+n) + a'\dot{p}(+tn) + a''t,$$
$$x = 2kp(t+m) + 2k'\dot{p}(t+m) + 2k''t,$$

in which u = supply of consumers' goods as approximated by these formulas,

x = amount spent on these goods,
p = retail price level of finished products,[a]
$\dot{p}$ = average quarterly change of p, and
t = time.

All variables are measured as deviations from their average value over the period considered, with this average value as a unit (time unit = one quarter of a year, $\dot{p}$-unit corresponding to price and time unit). Coefficients are regression coefficients (least square method) divided by correlation coefficients r.

[a] Calculations have been made for wholesale finished products prices; the coefficients found have been multiplied by the proportion between corresponding deviations from trend of wholesale prices and the cost of living index.

<div align="center">

A. UNITED STATES
</div>

n	Period covered by data for p	a	a'	r	m	Period covered by data for x	$2k$	$2k'$	r
−1	1921–1932	1.62	2.50	0.97					
0	1921–1932	1.73	2.18	0.97	0	1921–1932	1.20	4.37	0.79
		0.60	1.47					5.25	
0	1920–1932	0.72	4.41	0.74					
0	1920–1933	0.76	4.55	0.68					
0	1921–1933	1.67	2.18	0.96					
0	1922–1933	1.57	0.77	0.97					
1	1921–1932	1.83	1.82	0.95	1	1921–1932	1.64	3.12	0.81
2	1921–1932	1.80	0.94	0.92	2	1921–1923	1.94	1.21	0.84
3	1921–1932	1.61	0.04	0.94	3	1921–1932	1.95	−1.11	0.92

<div align="center">

B. GERMANY
</div>

n	Period covered by data for p	a	a'	r	m	Period covered by data for x	$2k$	$2k'$	r
0	1925–1933	0.96	3.51	0.90	0	1925–1933	1.85	0.84	0.96
1	1926–1933	0.96	2.17	0.92	1	1925–1933	1.87	−0.77	0.96
2	1926–1933	1.23	1.17	0.92					
3	1926–1933	1.41	−0.11	0.94					

A comparison with the long wave conditions for scheme D shows that, although these conditions are seldom exactly fulfilled, the discrepancies are often within the (rather large) limits of uncertainty in the coefficients.

Lag Patterns.—The verifications discussed so far are of a rough nature. It will be found that many alternative forms differing, for example, only in the fact that some small lags have been introduced lead to the same long wave conditions. For a more detailed verification, the lags themselves should be verified statistically. Attempts in this direction have up to the present yielded only meagre results, chiefly because of the influence of irregular movements and the lack of exact data as to different stages of the process of production. To show some
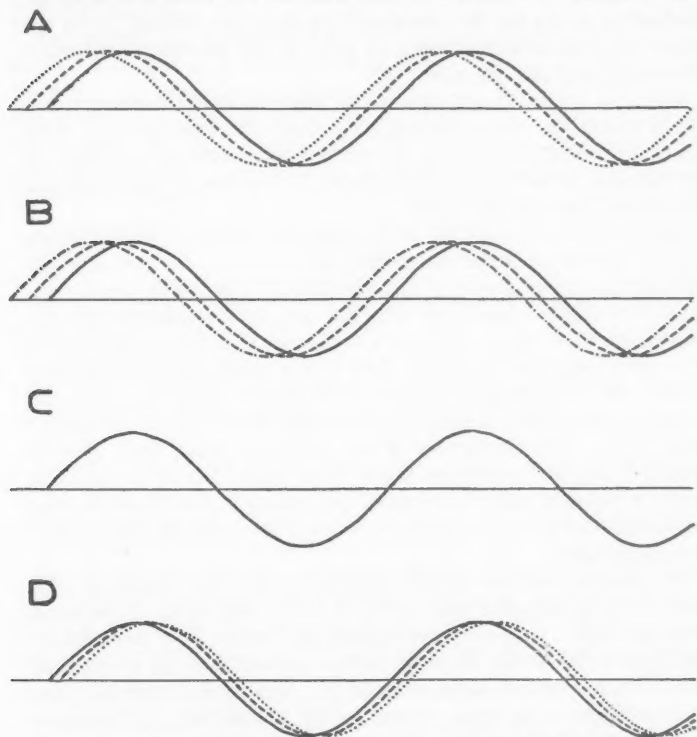
FIGURE 5.—*Lag patterns for some simple business cycle schemes.* (A for $\epsilon' = 1$, B for $\eta = 1$, C for $b = 2$ and D for $n > 0$) ······ Prices, —— Income spent on consumables, —— Supply of consumables, -·-·-·- Consumables sold. Curves that have not been drawn coincide with the supply curve.

of the details which I have in mind here, lag patterns have been drawn in Figure 5 for Cases $A-D$, indicating the phase differences that follow from each scheme for some of the important variables. The statistical observation of these phase differences is the difficult point; the large number of perturbations occurring in reality entails, in most cases, considerable changes in the order in which different variables pass through a given phase.

Sensibility of period to variation of constants.—One common property of the schemes discussed here must be emphasized. The period is very sensitive to changes in the constants when it is of the length corresponding to the one observed in reality. Although it may be recognized[71] that business cycles have periods that are far from constant, the variability shown by the periods of the short lag schemes seems too large. This circumstance seems to point to the necessity for further amendments of the schemes.

A similar argument against short lag schemes is that an interruption of the upward movement during a relatively short period (e.g., three times the production lag adopted) will not, in general, be followed by a continuance of the original movement. This seems to have occurred sometimes, however, in reality.

Asymmetry of waves.—One of the problems not solved by the simple schemes presented is the asymmetry of waves, i.e., the fact that the upward movement is generally slower than the downward. Apart from special speculative influences that fall outside the schemes hitherto considered, this may be attributed to the fact that the production lag is larger during the expansion than during contraction. In the first period, there will be situations in which stocks of raw materials and of semi-manufactured products are not sufficient to extend production immediately. The production of raw materials must, then, first be extended and the production lag in these stages has to be added to that for the stage under discussion. In a phase of downward moving production this is not true, since the lag will then only be equal to that of the last stage.

18. *Long lag schemes.*—In the schemes meant here, dynamical relations of a lag character play a rôle in which lags occur of the same order of magnitude as the wave lengths themselves. They have chiefly the significance of life times of certain objects such as investment goods. They interpret business cycles in a way more or less like this: suppose there is, for some reason, a high production of investment goods in certain years; then, the long life time of investment goods

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[71] As is stated by Hamburger, "Analogie des fluctuations économiques et des oscillations de relaxation," *Indices du mouvement des affaires*, Paris, ix (1931), Supplément.

will entail a sub-normal production of these goods in years to follow. When, in addition, there is a certain tendency among these life times to a grouping in the neighborhood of a given period, after this period there will be a revival of investment goods' production. This interpretation appears in the theories of Marx, Tugan Baranowski, Schönheyder, Cassel, Robertson, and De Wolff, but not as a closed mathematical scheme. I have attempted this for the simplest case, viz., when all investment goods are supposed to have one and the same constant life time.[72] The result is that, without any additional dynamical element, cyclical movements can be generated, the period of which is closely connected with the given life time. It need not, however, equal that life time, and cases are possible where it is only about ⅚ of that constant.

This simple case corresponds to the simplest possibility for replacement, mentioned in 4. Corresponding to the more complicated cases treated there, generalized schemes are possible. First, it may be supposed that the economic situation has influence on the moment of replacement (the case considered in 23), and, second, the assumption may be made that there are different sorts of machines each having its own life time. Details of this case have been treated by Frisch.[73]

Starting with a discontinuous frequency distribution of the different durations of life over the total number of machines, he shows that, after the lapse of a certain time, the movement of reinvestment as a function of time becomes very irregular when one considers each year separately (Cf. Figure 6). When moving averages are taken, however, a smoother curve is obtained which shows damped oscillations. The cases with a continuous frequency distribution, analyzed in a very elegant way by Frisch, also lead to damped oscillations.

Frisch assumes that the different sorts of machines in no way compete, viz., that neither do they themselves compete in the production of the same finished product nor do their products, if different, compete in consumption. Introduction of such possibilities of substitution have the tendency to restore a purely periodical movement.

As a fourth case, we have individual random differences in life time. A ten-year-old machine, like a five-year-old, may be replaced by a machine whose life time has a given probability distribution, but no given value. This case has been treated by Lotka, Kurz, and Vos,[74] who

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[72] *Loc. cit.*, p. 315.

[73] Mimeographed lectures, 1933, 852.

[74] Alfred J. Lotka, "The Progenity of a Population Element," *American Journ. of Hygiene*, Nov. 1928. "The Spread of Generations," *Human Biology*, Sept. 1929. Edwin B. Kurtz, *Life Expectancy of Physical Property*, New York, 1930. Vos, "Conjuncturcyclus en Techniek," de *Social. Gids* 1934, p. 464, and a paper to be published later.

reached the conclusion that, depending on the frequency distribution of life times, a movement with some cycles, or one without any cycle, may result.

The theories showing life times of replaced machinery as the only dynamic element or, more precisely, as the chief dynamic element have often been criticized. This criticism is usually of a negative and incomplete character. With regard to the statistical data in this field, the following information is now at our disposal: (a) rather rough data on depreciation rates used in practice and giving only average values, not
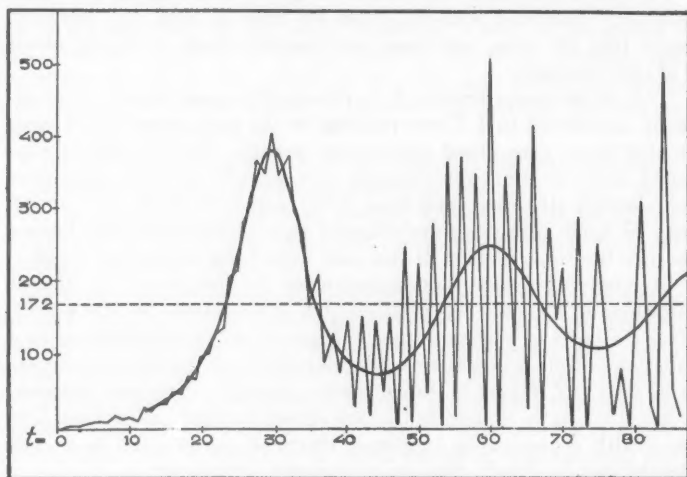
FIGURE 6.

frequency distributions, and (b) data on estimated technical and economical life time by producers of machinery; these are also averages, of course, but are classified according to categories of machinery. Vos gives a selection of data from the *Handbuch der Rationalisierung*, (published by the Reichskuratorium für Wirtschaftlichkeit, 1930) and from *Abschreibungssätze für Anlagen in Maschinenfabriken*, (published by the Verein Deutscher Maschinenbau-Anstalten, 1930). Most of the data relating to prime movers show a technical duration of life of about 20 years, an economic one of 8–15 years. For other machinery (textile, paper, metal industries), data are given only on the economic duration of life, averaging also around 8–15 years. (c) Some actual frequency distributions for automobiles (*Quarterly Journal of Economics*, Feb. 1933) show the following distribution:

Duration of life:
  0–1 1–2 2–3 3–4 4–5 5–6 6–7 7–8 8–9 9–10 10–11 11–12 12–13 13 years
Percent of cars:
  1   2   3   5   8   11   16   12   12   12   8   6   4   1

Mills (*Statistical Methods*) gives the following figures as to telegraph poles:

Duration of life:
  0–2 2–4 4–6 6–8 8–10 10–12 12–14 14–16 16–18 18–20 20–22 years
Percentage of poles:
  2   10   14   15   16   15   11   7   6   3   0

A frequency distribution of railway rolling stock in Holland was given in *De Nederlandsche Conjunctuur* (Aug. 1934). Some particulars are given below:

Life time in years:		Median	Quartiles
Locomotives	(437)	49	45;56
Passenger cars	(1030)	51	45;58
Freight cars	(5408)	52	33;59[a]
Luggage cars	(344)	49	43;59

[a] This distribution shows two maxima at 32 and 60 years.

The total number of each category included is given in brackets. It is obvious, however, that these data are highly incomplete.

One serious objection remains against theories that consider replacement waves as the chief factor in business cycles in that they neglect *new* investment too much. The only attempt to make a more or less complete estimate of replacement and new investment is that of the Berlin Institut für Konjunkturforschung,[74a] which gives the following figures (in billions RM):

Value of	1924	1925	1926	1927	1928	1929	1930	1931
Replacement	5.2	5.8	6.2	6.6	7.0	7.0	6.8	6.5
New investment	2.7	4.5	5.2	7.2	7.3	7.0	4.0	

There is one advantage in these theories, the larger stability of period in the face of changes in constants. Relatively large changes in the constants appearing in these schemes have only moderate influence on the period. The influence of life time itself is, of course, about proportional. But that influence is not large in comparison with the very large influence of some constants in the "short lag schemes" on the period (Cf. 15).

The ideal solution seems to be, therefore, the combination of short lag and long lag schemes in generalized types.

[74a] *Konjunkturstatistisches Handbuch*, 1933, p. 48.

SECTION IV. THE CHARACTER OF THE MOVEMENTS

19. *"Active"* and *"symptomatic"* variables.—In this section I give some general remarks on the quantitative behaviour of solutions, not relating to specific schemes discussed in section III but holding true for a rather large class of schemes. As examples, I shall choose the schemes mentioned, but the reader will realize that the statements are of a more general validity. It makes no difference whether we have a system of dynamic equations (irreducible to one single equation) or only one dynamic equation, after some substitutions, of course.

The first question is the different nature of the variables, which may be called active and passive (or symptomatic). By the latter, I mean variables that appear in only one equation of the original system (before substitution, etc.) and which are, therefore, automatically eliminated by omitting that equation; the latter is simply the definition equation of the variable in question. The economic significance of this mathematical property is that the variable in question is itself determined, or, to express it in a more suggestive way, dragged along, by the cyclic movement but does not reinfluence that movement. Mathematically, there is an interesting difference between the two groups of variables. The degree and, more generally, the functional form in which the active variables occur in their equations has a large influence on the nature of the movements of all variables. The form in which the symptomatic variables occur has influence only on their own movements. Suppose, for example, that in the irreducible system of equations all variables and their derivatives or integrals (relative to time) occur linearly. Then, the movements which these variables can show have the property that they may be multiplied by an arbitrary constant. Small movements owing, say, to small shocks and large movements due to large perturbations are strictly proportional. Moreover, the solutions are sums of very simple mathematical functions, that is, exponentials and damped sines. When the active variables would have appeared in, say, a second or third power or in transcendental forms in the irreducible system, the solutions are far more complicated, as will be shown later. It does not matter, however, whether symptomatic variables appear in such forms or not. Suppose one variable B to be symptomatic in some scheme and linked to one of the active variables A by some equation $B = f(A)$. The function may be such that, when A describes a sine curve, B describes a curve with flat minimum and a very pointed maximum, which seems a closer approach to reality for money rates, for example.

The rôle of active and symptomatic variables is, perhaps, best illustrated by considering the construction of a theoretical scheme in its successive stages. The theorist begins with some equations in which

a number of variables appear. The number of variables will at first, generally be larger than the number of equations. He must proceed until these numbers are equal, otherwise the system is not closed. We shall assume that he sets aside such equations as merely introduce one new variable and are thus to be considered only as definition equations for these variables, being themselves "symptomatic." When the system is closed after only a small number of active variables has been introduced, this means that a small group of phenomena is chosen from the economic total and their interrelations are considered to be the causes of the cyclical movements, the other variables being forced to move in dependence on them. This is more or less the case in all simpler schemes. The remarkable thing is that it is possible to choose quite different phenomena in this small circle. However, the more realistic the schemes are, the larger the circle of active variables it is necessary to introduce.

20. *Equilibrium points and small movements.*—An important preliminary step in the solution of the equations is the determination of the equilibrium points, i.e., the *constant* values of the variables that satisfy the equations. Substituting constant values for variables entails a substantial simplification of the equations, as all derivatives with respect to time vanish, all integrals reduce to linear terms, etc. A system of linear equations will, of course, lead to only one equilibrium point (provided its determinant is not zero), but non-linear equations may yield more. Thus, Kalecki's equation shows only one point, as does the simplest case I considered. Some of my more complicated schemes, however, show two equilibria and it is easy to construct schemes that show many. Vinci's system, however, yields only one equilibrium solution.

The equilibria found may, or may not, be *stable*. The criterion is the nature of small movements around the equilibrium. These small movements may be divided into three groups: (1) movements directed, in the long run,[75] away from the equilibrium; (2) movements directed, in the long run, towards the equilibrium, or oscillating movements with positive (or zero) damping; and (3) oscillating movements with negative damping (anti-damping).

The equilibria relating to the first type of small movement may be called unstable, those relating to the second, stable, and those to the third group, quasi-stable.

Whether an equilibrium belongs to a certain group depends, in general, on the special values of the constants. For some values it

[75] This expression means that a finite time can be indicated after which the movements will continue in the direction shown without interruption.

may be stable, for others unstable or quasi-stable. Thus, in Kalecki's case, an unstable equilibrium exists when

$$e^{-m}(m + \theta n) \leqq \frac{1}{e},$$

in which m, n, and θ, are his constants. In the case discussed in 15A (for $\epsilon' = 1$), an unstable equilibrium for $p = 0$ exists for

$$\theta k > \frac{aP}{A} + 1,$$

where, in the case $aP/A = 1$, equilibrium is stable when

$$\theta k \leqq 2.$$

It is interesting that, in this last case, one equilibrium is stable and the other unstable and that the stable one is the one with the highest value for the variable p (price).

Let x, y, etc., be the variables of any dynamic set-up. We choose the origin of our system of variables at the equilibrium point, i.e., in equilibrium we have $x = 0$, $y = 0$, etc. Then x, y, etc., are simply the deviations from equilibrium shown by our variables. It follows that our equations do not contain any constant terms but only terms with one or more factors, x, y, $\dot{x}$, $\dot{y}$, $\int_{t-\epsilon}^{t} x \, d\tau$, or $x(t-\theta)$, etc., where t is time, (τ, when integration variable), ϵ and θ constants. If we also had constant terms, the equilibrium point would not satisfy our equations.

Since we now suppose the movements to be small, all terms containing products of two or more variables (derivatives, integrals, or "lagged variables," included) may be neglected. Thus, we have to study only equations that are linear in these variables, in the wide sense of the word.

This being so, the nature of the components of the movement will be determined by the roots of a characteristic equation. If one of these roots has a positive real part, unstable equilibria will exist. Stable equilibria occur if none of the real parts are positive. If all the roots are actually real and non-positive, then we will have a movement, in the sense mentioned above, directed in the long run towards equilibrium. When there are complex roots, we will have oscillations of decreasing amplitude (positive damping), as soon as all moduli are below unity.

Quasi-equilibria will occur when there are any complex roots having a modulus larger than 1. This entails that the system show at least one oscillation whose amplitude increases beyond any limit as time passes.

This sort of oscillation is impossible in physics as far as endogenous oscillations are concerned, as they are incompatible with conservation of energy. As such a law does not necessarily apply to all economic schemes, they are there possible.

21. *Non-linear equations.*—New problems arise when the irreducible system or original equations is not linear. Then, the theorems concerning small movements around the equilibrium or equilibria are not valid for larger movements. A general discussion becomes, in most cases, very difficult. By examples I shall now show what sorts of new possibilities are then created.

In my paper on oscillatory movements in ECONOMETRICA (1933) p. 44, I discussed a case where it was possible, by introducing a new variable, to transform a non-linear into a linear one. The new variable $\omega(t)$ depends on the old one by

$$\omega(t) = \log \left[bp(t) + 1 \right].$$

One of the possible movements of ω is a sine curve of arbitrary amplitude. The corresponding movement of $p(t)$ is similar to the movement of the "symptomatic" variable considered in 19.

A more complicated case is treated in my paper in the Wagemann *Festschrift*, in the sense that some numerical calculations have been made. These calculations show that, for the same set of constants, the period of the movement changes with the initial condition. For instance, when the initial impulse is assumed to be increasing, the period increases, reaching infinity for a finite value of the initial disturbance. Parallel with this growing period, the form of the oscillations is changed. When the movement comes near the second, unstable equilibrium that exists in this case, it tends to move for a long time parallel to that equilibrium and so to prolong the part of the oscillatory movement situated between the two equilibria. When the original disturbance surpasses a certain limit, however, the movement breaks through the unstable equilibrium and the movement does not come back.

It should be stressed that only the first beginnings in this field have been made. On the mathematical side of the problems, Volterra's book, *Sur la théorie mathématique de la lutte pour la vie*, may be mentioned.

I restrict myself to economic aspects. First, it may be stated that higher degree equations are very likely to occur in economics. The simple relation between price, quantity sold, and amount paid, is a quadratic relation. As the number of variables included increases, there are many other reasons for obtaining higher degree equations.[76]

There is one rather general phenomenon leading to non-linear func-

[76] Koopmans, "De mogelijkheid van meervoudig evenwicht," *De Economist*, 1932, pp. 679, 766, and 841, has shown this in an elementary way.

tions in our equations, the phenomenon that might be indicated by *saturation*. Its essential feature is that certain variables are not allowed, by natural, technical, or institutional circumstances, to surpass a given limit. An example is given in the case where a maximum output exists for technical reasons. Assuming, for simplicity, that this output depends only on price, the relation between price and output will be non-linear; output as a function of price will have a horizontal asymptote. The simplest case presents itself, of course, when the saturable variable is a symptomatic one, whereas the active ones only appear in linear equations. We then have the following situation. When the intensity of the initial perturbancy grows, the active variables show amplitudes proportional to this intensity but the symptomatic ones show amplitudes that, after a certain limit in the intensity has been reached, grow slower and slower, the saturation limit being its limit. In an analogous fashion, the form of the waves in these symptomatic variables is different from that of the active variables, and these waves will be flattened.

When active variables also show saturation, we have a special case of non-linear irreducible equations. The question has not yet been investigated in this way, although the phenomenon as such is often mentioned, even in literary theories. I can only refer to one aspect. It is a source of rather frequent misunderstanding in this field that, when the saturation limit is reached, this would mean at the same time that the turning point, in the sense of the point where the downward movement begins, is reached. It is possible to give a very simple example where this does not hold good. It is the case of the simple "Hanau oscillation," showing inverse waves of prices and deliveries.[77] When, in this scheme, one of the variables has a maximum prescribed by technical conditions, we may have a situation where the maximum is maintained a certain finite time before the downward movement begins.

22. *Some further problems regarding maximum amplitude.*—The following are some further suggestions regarding saturation and maximum amplitude. We know by experience as well as by theoretical considerations that a large number of different activities develop with a certain parallelism during the upward swing of the wave and shrink, in about the same way, during the downward movement. Of course, the variables themselves need not be proportional, but their deviations from equilibrium may be approximately proportional. What will now determine the maximum amplitude of the whole system? That activity whose amplitude limit is reached *first*. The limit may be not so sharply defined; it may be somewhat elastic, but there will be a limit beyond which, for example, in the short run, production of pig iron, or the amount of loans, or the number of workers, cannot be extended. The

[77] Cf. my paper in ECONOMETRICA I (1933), 38.

limit will be a wider one for one phenomenon than for the other and one phenomenon will have the tendency to more expansion than the other, and so forth. How can it be determined statistically which category will first reach its limits? We can say, as a first approximation, the good or service whose price shows a sharp peak, whereas other prices, and especially the output or amount used of the good or service itself, only show flat peaks. As examples, I see pig iron prices before some of the earlier crises (1873, 1890, 1900), discount rates in nearly all crises, and, in some cycles, stock prices (1929) or freight rates (1900). As examples of the opposite, we can mention wages and prices of agricultural products. For this reason, I see more realism in theories that point to a scarcity in capital goods or in short credit than in theories pointing to a scarcity in labor or organic materials.

SECTION V: COORDINATION OF THEORIES

23. "*Alternative dynamizations.*"—The existence of a rather large number of competing theories is not a satisfactory situation. It would be most desirable to have one generally accepted explanation. The way to approach the problem is, first, to confront the theories and, second, to combine their really valuable parts. To carry such a synthesis through is no easy task. In this synthesis there are two different phenomena which I should like to call (a) alternative and (b) different schemes and dynamizations. By alternative schemes I mean schemes or theories pretending to represent the same phenomena but doing so in a different way. By different schemes I mean schemes representing different phenomena, though, of course, they pretend to represent an approach to the same thing. For example, Kalecki's and my schemes both pretend to represent an approach to a society showing business cycles, but, in Kalecki's scheme, elementary phenomena are included other than in mine. He introduces an influence from total capital on amount invested, while I introduce an influence of prices on amount invested. In this respect these schemes are, therefore, different schemes.

First, consider alternative schemes or, as the principal characteristic of the schemes is the "dynamizations" in it, alternative dynamizations. Taken in the true sense of the word, they are few. An example is the two ways in which variation of wages may be connected with variation in employment by a simple lag equation or by a differential relation. Another example is the adaptation of different "time shape functions of production," partly the controversy between Hayek and myself as discussed in 9. A third is the rules governing replacement, the controversy between (a) Kalecki, assuming constant replacement, (b) De Wolff, assuming replacement varying according to age of objects, and (c) others, assuming replacement varying with the economic

situation. It is clear that such questions can only be settled by statistical evidence.

Somewhat different questions may be treated in this category, viz., questions of *different degrees of approximation*. Can the supply function (cf. 3, 15), for example, be assumed as linear or should a higher degree function be chosen? Can a constant lag be assumed between price and delivery, or should it be chosen variable? Can a fixed life time be assumed for means of production or should a variable life time be taken? Although, in these questions also, statistical evidence must be decisive as to the necessity of a certain higher approximation, theoretical analysis can inform us on the character and, in some cases, the direction of the "corrections" that such higher approximations entail. Some examples may be given here.

E.[78] *Influence of small lag variations on Kalecki's scheme*

Suppose θ (12) is no longer constant but depends on some variable x (to be chosen later on) by the equation $\theta = \theta_0 + \theta_1 x$. This will change the equations given in 12. The first of these equations, (12.1), showing the relation between $L(t)$ and $I(t-\theta)$, will be changed due to the following circumstances. Total investment orders, $I(t)$, at time t correspond to deliveries, $L(t+\theta)$, at time $t+\theta(t)$. Investment orders, $I(t+1)$, at time $t+1$ correspond to deliveries $L(t+1+\theta(t+1))$. Whereas one time unit elapses between the orders, between the deliveries $1+\theta(t+1)-\theta(t)$ $=1+d\theta(t)/dt$ units of time elapse. According to the value of the last term, there will have taken place a "tightening," or an "evacuation," of orders and, therefore, we have, instead of (12.1),

$$L(t + \theta) = \frac{I(t)}{1 + \dot{\theta} \cdot (t)},$$

or, as for $\theta(t)$ we can write $\theta_1 \dot{x}$,

$$(1 + \theta_1 \dot{x})L(t + \theta) = I(t).$$

No change will occur in equation (12.2), but equation (12.3) may be replaced by

$$W = \int_t^{t+\theta} L(\tau)d\tau,$$

in which θ is now a variable. This entails that

$$\frac{dA}{dt} = \frac{d}{dt}\frac{W}{\theta} = \frac{\dot{W}}{\theta} - \frac{W}{\theta^2}\dot{\theta} = \frac{\dot{W}}{\theta} - \frac{W}{\theta^2}\theta_1\dot{x}$$

$$= \frac{L(t + \theta) - L(t)}{\theta} - \frac{\int_t^{t+\theta}L(\tau)d\tau}{\theta^2}\theta_1\dot{x}.$$

[78] A-D have been given in 15.

Equation (12.4) of 12, written in differentiated form,

$$\dot{I}(t) = m\dot{A}(t) - n\dot{K}(t),$$

can now be transformed with the help of:

$$L(t) = L_0 + L_1(t); \quad \dot{I} = (1 + \theta_1\dot{x})\dot{L}(t + \theta) + \theta_1\ddot{x}L(t + \theta)$$

$$\text{and} \int_t^{t+\theta} L(\tau)d\tau = \theta L_0 + \int_t^{t+\theta} L_1(\tau)d\tau,$$

in which L_0 represents the equilibrium value of $L(t)$, and $L_1(t)$ is supposed to be small. When we choose for x,

$$x = \int_t^{t+\theta} L_1(\tau)d\tau,$$

i.e., when we assume that the production lag depends on activity, we find after some computations that Kalecki's equation is replaced by a new one, obtained by putting

$$\theta_0(1 - \theta_1L_0) \qquad\qquad \text{instead of } \theta,$$

$$\frac{n}{1 - \theta_1L_0} \qquad\qquad \text{instead of } n,$$

$$m(1 - \theta_1L_0) - \theta_1L_0 \qquad \text{instead of } m.$$

F. Influence of small variations of life-duration on the scheme referred to in 18 (Note 72)

As a second example, I treat the influence of small variations in life time of means of production caused by changes in the economic situation, basing myself on a scheme given in my *Zeitschrift* paper. In this scheme, the economic situation is characterized by the price level of capital goods q, and the only thing to be varied is that, instead of supposing the duration of life to be constant, we now assume it to depend on q in the following way:

$$\eta = \eta_0 - \rho q(t), \tag{23.1}$$

where η may now mean the life time of means of production replaced at time t.

Our equations (given in *Zeitschrift f. Nationalokonomie*, 1934, p. 316)

$$\int_{t-\eta}^t q(\tau)d\tau = gq(t), \qquad g = \frac{a}{b}\,\frac{e + f - A}{A + aP - e}, \tag{23.2}$$

must now be changed in accordance with (23.1). In the first of equations (23.2), at both sides a constant term has been omitted, viz., the

terms $\int_{t-\eta}^{t} Q d\tau = \eta Q$ at the left side, and some constant K at the right side, that is equal to ηQ by virtue of the long run equilibrium condition. As, however, η is now no longer a constant, we have to reconsider this term. It will be replaced by

$$\eta_0 Q - \rho q(t) Q,$$

and our equation becomes

$$\int_{t-\eta}^{t} q(\tau) d\tau = (g + \rho Q) q(t).$$

The effect is simply that g is replaced by a somewhat larger g. The influence of this larger g on the period depends on g's value. For $g = 1$, a minimum period is reached, for $g < 1$, an increase means decreased period, but for $b > 1$, an increase means an increased period; as to r, which measures inversely the degree of damping, this moves contrary to g, and so an increase of g means an *increase* in *damping*.

Mathematical approximation.—Apart from questions of different degrees of approximation to relations that must be found empirically, theoretical relations may also be approximated in order to facilitate calculations. We have already met examples of this sort. Integrals over short intervals—in comparison with periods or quasi-periods—may be replaced by a mean value times interval length, integrals over longer periods by sums, derivatives by differences, etc., and *vice versa*. So it will be possible, for example, to get rational, instead of irrational or transcendental, equations.

Another possibility consists in a reduction of all lagged variables to derivatives, by

$$f(t - \theta) = f(t) - \theta f'(t),$$

which holds, of course, only for small θ's, that is, small as compared with the period of the movements considered. In this way, it is possible to reduce a large number of equations to *pure differential equations*. Of the last possibility, an example is given in the next paragraph.

24. *"Different" dynamizations.*—By far the most important difficulty we encounter when we desire to coordinate theories is that of the confrontation and combination of "different" dynamizations. We have seen in Section II that there is a large number of dynamic relations among the most important economic variables. The number of alternatives is so large that it is virtually impossible to take account of all of them. This has lead several authors to make a choice and the choices have been different. One is lead to ask what choice was best? But the question is not posed well. None is absolutely right or absolutely wrong. Progress is not to be sought through a choice but a combina-

tion. Our aim should always be to reach conclusions of the following kind: "when these coefficients are small and those large, author A gives a better approximation; when those are small and these large, author B does. When all coefficients are of importance, a mixed theory emerges deviating from author A's theory by this and from author B's theory by that peculiarity."

G. A combination of some principles of Frisch's and mine

As a very simple example, I combine Frisch's theory on the interrelation between consumers' goods and capital goods production with my method of introducing prices into the schemes. Let the production of consumers' goods be $x(t)$, and that of capital goods $y(t)$; then, according to Frisch, we have:

$$y(t) = mx(t) + \mu \dot{x}(t). \tag{24.1}$$

The production of consumers' goods itself may depend on the price level of consumers' goods, $p(t)$, and be, for example,

$$x(t) = bp(t) + c\dot{p}(t). \tag{24.2}$$

Apart from a constant factor of proportionality, the right member in (24.2) may be looked upon as an approximation to the price level existing at some time a little before t when $c < 0$ (and $b > 0$), or a little after t when $c > 0$. A *lag* of x after p (for $c < 0$) may be caused by the length of the production period; a lag of p behind x may be caused by the fact that producers determine production by paying attention to probable further price development or by the fact that it is not the price level that determines production but production (as dictated by consumption) that determines prices.

The price level itself will be determined at the consumers' goods markets, for which we have

$$(P + p)(X + x) = K + L(x + y). \tag{24.3}$$

Here, according to our usual notation, P, X, and Y, denote equilibrium values, L the average income earned by additional producers for each unit of additional product. Limiting ourselves again to small movements (small p, x, and y) and choosing units such that $P = X = 1$, we have,

$$p + x = L(x + y). \tag{24.4}$$

The solution of our system, (24.1), (24.2), and (24.4), gives us

$$p + x = L(x + mx + \mu \dot{x}).$$

Writing λ for $L(1+m) - 1$, which will, in all probability, be positive, we have

$$p = \lambda x + \mu L \dot{x},$$

and thus,

$$p = \lambda b p + (\lambda c + \mu L b)\dot{p} + \mu L c \ddot{p},$$

or,

$$\gamma p + (\lambda c + \mu L b)p + \mu L c \ddot{p} = 0. \qquad (\gamma = \lambda b - 1).$$

This is the equation for a harmonic oscillation, if

$$4\gamma \mu L c > (\lambda c + \mu L b)^2.$$

I shall not attempt a full discussion but only state that no oscillations at all are possible when γ and c have opposite signs and that, in this example, another form of equation is obtained into which many combined schemes can be brought by suitable approximations. A fuller discussion and further attempts to generalize this scheme should be postponed until better statistical information is available and some of the controversies discussed in Section II are clarified.

In this connection it is important to remember that "different theories," in the economic sense of the word, may often lead to the same mathematical formulas, the only difference being in the economic sense of some of the constants. I have already pointed out, for instance, that the influence of wages and productivity of labor on profit margin w are both proportional to

$$- \int_0^t p(\tau)d\tau \qquad\qquad (\text{cf. 5}).$$

The same might be said of the influence of the total capital invested on profit calculations, i.e., when entrepreneurs are led by the percentage profits bear to this capital. Then the directive of production will be

$$\frac{W + w(t)}{C + c(t)},$$

W and C meaning equilibrium values of profits and capital and $w(t)$ and $c(t)$ being deviations from equilibrium values of these variables. This expression may be approximated by

$$\frac{1}{C} \left\{ W + w(t) - \frac{W}{C} c(t) \right\}.$$

The last term in the brackets will be proportional to $\int_0^t p(\tau)d\tau$, as the *growth* of capital is proportional, in its cyclic component, to $p(t)$. Thus, this last term is of the same structure as the productivity and the wages term.

SECTION VI. PROBLEMS OF BUSINESS CYCLE POLICY

25. *Consequences of a given policy.*—The ultimate practical purpose of all business cycle theory and business cycle research is, of course, to help solve the problems of business cycle policy. The theoretical nature of these problems is that of variation problems; each case of business policy may be looked upon as a variation of some constants of a business cycle scheme. In this respect, they belong in the same class as the problems depending on the influence of other ("natural") perturbances of cyclical movements, such as exceptional crop figures. Some of these problems are analogous to "shock problems" in pendulum physics; another part may be compared to the problem of changing the length of a pendulum. A third category consists in a shifting of the turning point of the pendulum.

In mathematical terms, this is about equivalent to the distinction between three sorts of variation problems, (a) variation of the initial conditions, (b) variation of the coefficients of one or more of the equations, and (c) variation or introduction of an additive term in one or more of the equations of the system.

Apart from this distinction, another can be made, that between "single" and "composed" variations. By a "single" variation, a transition from one given value of a constant to another is meant; by composed variations, a system of consecutive changes in one or different constants.

Some examples will illustrate these notions.[79] Three types of variation problem may be treated: (I) Influence of varying crops on cycles; (II) Influence of "money injections"; and (III) Influence of wage changes, as far as these are not endogenous.

To study the consequences of given economic types of perturbations, one has to choose a suitable scheme.

In our examples, I take as the undisturbed scheme the one discussed in 15, A for $\epsilon' = 1$, i.e., the simplest pure lag scheme. Using $p(t)$ as variable, the supply of consumers' goods at time t is $A + ap(t-2)$. The amount spent on the consumers' goods market at time t is

$$K + 2kp(t-1),$$

and the equation determining the movement of $p(t)$ will be:

$$\{A + ap(t-2)\}\{P + p(t)\} = K + 2kp(t-1). \quad (25.1)$$

In this equation we will now consider the three variation problems mentioned.

[79] As I shall point out, problems of type a and type c are reducible to each other; single variation problems in class c, for example, can be transformed into composed variation problems in class c. Other transformations also exist.

I. Influence of variations in crops

Variations in crop yields have as a consequence that supply consists of a "regular" part, $A + ap(t-2)$, cf. equation (25.1), and an irregular one, for which we may write $A'(t)$. At moment t we have, therefore, instead of (25.1)

$$\{A + ap(t - 2) + A'(t)\}\{P + p(t)\} = K + 2kp(t - 1). \qquad (25.2)$$

At each moment, this means that the "new" price, $P + p(t)$, is not determined by the "regular" equation (25.1) leading, with small p's, to a sine curve, but by a somewhat changed equation (25.2) which may be looked upon as the regular equation corresponding to another value of $p(t-2)$, viz.,

$$p'(t - 2) = p(t - 2) + \frac{A'(t)}{a}. \qquad (25.3)$$

In the case of *small* movements, we may say that the price movement all at once follows another sine curve and such a changing of the curve followed presents itself each time $A'(t) \neq 0$. Considering the problem in this way, we have to do with a variation in initial conditions. Of course, it may also be looked upon as a combination of a variation in additive constants and coefficients. This leads to the creation of changing "harmonics" by erratic shocks, according to Frisch's scheme.

II. Influence of "money injections"

When, at moment t, an amount $K'(t)$ is added to the amount spent at the consumers' goods market (e.g., when public works are paid for with created money), we get

$$\{A + ap(t - 2)\}\{P + p(t)\} = K + 2kp(t - 1) + K'(t). \qquad (25.4)$$

This can be interpreted by saying that the price, not at moment $t-2$, but at the moment $t-1$, has changed, the amount of variation being

$$\frac{K'(t)}{2k}.$$

In accordance with expectations, a positive increment of $p(t-2)$, as in problem I, has a negative influence on $p(t)$, whereas a positive increment of $p(t-1)$, as in problem II, has a positive influence on $p(t)$.

Restricting ourselves to small movements, in which case we get linear equations, we can interpret the influence of the variation $K'(t)$ also as a changing of the equilibrium level of prices, formerly P. Indeed, we can write for

$$Ap(t) + aPp(t - 2) = 2kp(t - 1) + K'(t), \qquad (25.5)$$

equivalent to (25.4), the equation

$$A\{p(t) - p_0\} + aP\{p(t - 2) - p_0\} = 2k\{p(t - 1) - p_0\}, \quad (25.6)$$

when we take

$$(A + aP - 2k)p_0 = K'(t). \quad (25.7)$$

This means that there are two ways in which $p(t)$ can be calculated, first, as the value belonging to $p(t-2)$ and $p(t-1)+K'(t)/2k$ in the old mechanism and related to the old equilibrium level and, second, as p_0 plus the value belonging to $p(t-2)-p_0$ and $p(t-1)-p_0$ in the old mechanism, but related to a new equilibrium level $P+p_0$.

In this case of small movements, we can even give two further interpretations; the additional term may equally well be considered as a variation of $-K'(t)/A$ in $p(t)$ as a variation of $-K'(t)/aP$ in $p(t-2)$.

When, in only one elementary time period (i.e., a period of length 1), an "injection" takes place, we have, for a succession of periods, the equations:

$$Ap(2) + aPp(a) = 2kp(1)$$
$$Ap(3) + aPp(1) = 2kp(2)$$
$$Ap(4) + aPp(2) = 2kp(3) + K' \quad (25.8)$$
$$Ap(5) + aPp(3) = 2kp(4)$$
$$Ap(6) + aPp(4) = 2kp(5),$$

the period of injection being period 4. Considering the variation as a variation in $p(2)$, the variation is a "single" variation in initial conditions, i.e., the movement jumps but once from one curve to another. Considering K' as a variation in equilibrium price level, however, we have to do with two jumps, one after another. First, a jump from equilibrium level P to $P+p_0$ takes place and, second, a jump back to P. We have, then, to do with a "composed" variation of class c, which is evidently identical with a "single" variation of class a.

III. *Influence of wage changes*

As, in the original scheme adopted here, no wage changes occur, all changes in wages are exogenous. A wage change has two sets of consequences, first, costs of production and, therefore, production plans are changed and, second, the income spent at the consumers' goods markets is changed. The first consequence means that, after a wage change, p in the supply and income functions has to be changed to

$$p - p_0$$

and a and k into $a\alpha$ and $k\beta$; p_0, α, and β, depending on the change in wages adopted. The second consequence entails that labor incomes

and, therefore, incomes spent corresponding to a given productive activity have changed, i.e., K becomes $K\gamma$ and $k\beta$ again changes into $k\beta\gamma$. Thus, after the wage change has been worked out fully, equation (25.1) has turned into

$$\{A+a\gamma p(t-2)-a\gamma p_0\}\{P+p(t)\}=K\gamma+2k\beta\gamma\{p(t-1)-p_0)\} \quad (25.9)$$

which, in general, means a change in the mechanism, i.e., in period and degree of dampening, contrary to the other examples given. But this is not the only new feature. The new terms and factors occurring in (25.9) do not come in at the same moment, but at consecutive moments, the order of which may still be different owing to different assumptions. So, it is in every case clear that changes in supply will be later than those in incomes in this scheme. But, in addition, γ may or may not come in at the same time as β and p_0 come in at the right side. And this influences in a very important degree the shape of the disturbance.[80]

It will be seen that, in this third problem, we have to do with a composed variation of initial conditions as well as of coefficients.

Having considered these three variation problems, we can now state that, in general, a perturbation may change four fundamental aspects of the movement, the amplitude, phase, period, and degree of dampening. In common discussions on business cycle policy this is seldom recognized, for often only immediate effects are taken account of. Another statement of importance is that the effect often depends in a very large degree on the phase of the original movement in which the perturbation presents itself.

26. *The problem of the "optimal" policy; some remarks on "neutral" money.*—Having indicated, at least in principle, how the consequences of a given business cycle policy are to be found, we can now enunciate the "last" problem in this field, that is, which policy is the "best" one, the "optimal"? To answer this question, we should dispose of some general ophelimity function and calculate for which policy a course of events would occur giving a maximum to this general ophelimity function.[81] Of course, at this moment any practical calculations of this sort are impossible. I shall not try to give them here.

I shall only make some remarks on the frequent and, I think, justifiable assumption that stabilization of business cycles in employment is the optimum policy.

Stabilization of cycles can be obtained in different ways, viz., by

[80] Cf. my paper: "La politique des salaires, les cycles économiques et les mathématiques," *Revue des Sciences Économiques*, févr. 1935, p. 17.

[81] Cf. in this respect my tentative "Konjunkturforschung und Variationsrechnung," *Archiv fur Sozialwissenschaft und Sozialpolitik* LXI (1929), 533.

(1) reducing the amplitude of the cycles only; or (2) reducing the period of the cycles only; or (3) increasing the dampening of the cycles only; or, finally, by combinations of these types of policy. An example of the first form of stabilization is given by some cases of wage changes, examples of the second categories by some cases of "money injections"[82] or a "raw material monetary standard."[83] Of these three types, the first one is essentially less advisable than the other two, as a new perturbation of equilibrium will immediately change the amplitude and may make it large again. This is unlikely to occur in the second and third types, as a certain degree of continuity will also hold the amplitudes of the movements within limits of these cases. Thus the second and third method are to be preferred.[84]

Another difficulty is also evaded by the second and third types of stabilization policy, viz., the difficulty arising from the fact that the equilibrium value of prices and other variables continuously changes as a result of innumerable variations in data. A practical approximation to this equilibrium level by computation will hardly be possible before a highly organized economic control exists. By adopting, however, a business cycle policy of type 2 or 3, i.e., by changing our economic mechanism so as to show only very short or very much dampened oscillations, we could arrange it so that this equilibrium level would be realized automatically. I hope to add further contributions to this problem within a short time.

It should be borne in mind, however, that the examples of class (2) above relate only to highly simplified schemes and that they do not necessarily behave similarly for other schemes. The remarks made, therefore, are valid for these examples of policy only in so far as the simple schemes are considered. But they relate to many more general examples as well.

I may add a few remarks on a special form of stabilization policy that has been proposed and worked out by several authors during the last decennium, though not by all of them in the same way. This policy, often indicated as a neutral money policy, has perhaps been defined and defended in the most explicit and exact way by Koopmans.[85] To summarize his argument, the state of things which obeys the Walrasian system of equations is an optimal situation. As the equality of supply and demand (in terms of goods) is disturbed by the hoarding and creat-

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[82] Cf. the papers mentioned in notes 79 and 56.

[83] Cf. the paper mentioned in 24, Concluding remarks, 2.

[84] Cf. Frisch, mimeographed lectures, 1933. Frisch, in particular, advocates method 3. He calls it the principle of "the oil brake"; a brake is introduced whose strength is proportional to the *velocity* of the movement.

[85] J. G. Koopmans, "Zum problem des 'neutralen' Geldes", published in *Beitrage zur Geldtheorie*, herausgegeben von F. A. Hayek, 1933.

ing of money, the monetary system must be such that each hoarding and creating is counteracted by equal amounts of money created or hoarded. This should be reached by credit policies, i.e., discount and open market policies. Koopmans shows especially that a neutral monetary system is not equivalent to a constant price level monetary system.

This last statement, is, no doubt, valid. I should, however, like to make some remarks on the preceding argument. In the first place, the identification of the optimum situation with a Walras situation is, in my view, very questionable. Since it seems that Koopmans himself recognizes this, it may be left out of the discussion. My main objection is, then, that the realization of the Walrasian situation is impossible when we have a *permanently changing situation* in addition to some elements that make absolutely impossible an immediate reaction of all variables to any change in data. Elements such as the production period and the long duration of means of production make it impossible that a quick adaptation to new data should take place. Therefore, the Walrasian situation can only accidentally be realized. This is the reason why it seems better to discuss the desirability of a given stabilization policy without any connection with the Walrasian system.

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ON THE INDEPENDENCE OF k SETS OF NORMALLY DISTRIBUTED STATISTICAL VARIABLES

By S. S. WILKS

I. INTRODUCTION

IN SUCH fields of investigation as economics, psychology, and anthropology, where observations on several variables are taken into account simultaneously, it is at least as important to study relationships among the variables as to consider the variables separately. In fact, if there are significant relationships within a system of variables, a considerable part of the information furnished by the observations will be lost unless the relationships are taken into account. In general, very little is known *a priori* about such a set of variables, and hence our knowledge of them and their various interrelations must be inferred from observations. Questions relating to the problem of making inferences from observations resolve themselves into those of, (1) devising suitable functions of the observations for estimating parameters which characterize the hypothetical population of the variables and (2) determining frequency laws from which the degree of credibility to be placed in the departure of these functions from expectation can be evaluated. The more complicated the hypothesis concerning the interrelations of the variables, the more complex, of course, will be the functions of observations for measuring the relationships and testing the hypothesis. It frequently happens in multivariate analysis that a number of variables can be rationally classed into several mutually exclusive categories. For example, certain measurable traits of individuals may be classed as physical or mental. In the study of wholesale prices of farm products in a certain region over a certain period of time, the products may be classed as (1) fruits, (2) vegetables, or (3) dairy products, and the deviations of the prices of products within each group from seasonal and secular trends may be taken as the variables. When variables can be grouped in such a manner the question naturally arises as to whether or not there is any significant relationship between the groups of variables. That is, on the basis of the available observations, with what degree of credibility can we assert that the groups are mutually independent, so that knowledge relative to one of the groups gives us no significant information about the others? If they are significantly non-independent how can we measure the amount of dependence? It will become apparent as we proceed that statistical functions[1] and significance tests more general and comprehensive than

[1] See R. Frisch, "Correlation and Scatter in Statistical Variables," *Nordisk*

those used now will be needed to answer these questions satisfactorily. It is the purpose of this paper to consider the problem for the case of normally distributed variables.

To state the problem more precisely, suppose N individuals have been drawn at random from a statistical universe U of n variables $X_1, X_2, \cdots, X_n$ distributed according to a normal frequency law

$$(1) \qquad P(X_1, X_2, \cdots, X_n) = Ke^{-1/2\Sigma_1^n A_{ij}x_ix_j}$$

where $\|A_{ij}\|$ is a positive definite matrix with determinant A, $x_i = X_i - m_i$, m_i being the means of X_i and $K = \sqrt{A}/(2\pi)^{n/2}$. If σ_i^2 is the variance of X_i and ρ_{ij} is the correlation coefficient between X_i and X_j, then

$$A_{ij} = \frac{\Delta_{ij}}{\sigma_i^2\sigma_j^2\Delta}$$

where Δ is the determinant $|\rho_{ij}|$ and Δ_{ij} is the cofactor of ρ_{ij} in Δ. $\rho_{ij}\sigma_i\sigma_j$ is known as the covariance of X_i and X_j. Let the observed values of the variables of the αth individual be $X_{i\alpha}(i=1, 2, \cdots, n)$ and let the set of all observations be O. We shall refer to O as a sample. Now suppose we have reasons, depending upon the problem under consideration, for grouping the variables $X_1, X_2, \cdots, X_n$ into k groups $\{X\}_m(m=1, 2, \cdots, k)$ where $\{X\}_m$ denotes the variables $X_{\bar{n}_m+\mu}$ $(\mu=1, 2, \cdots, n_m; \bar{n}_m=\sum_{\beta=1}^{\beta=m-1}n_\beta)$, the mth group having n_m variables.

The problem can now be stated as follows: On basis of the evidence furnished by O, how can we test the hypothesis that the sets of variables $\{X\}_m$ are mutually independent? By mutual independence, we mean that $P(X_1, X_2, \cdots, X_n)=P_1 P_2 \cdots P_k$ where P_m is normal and independent of all variables except those contained in $\{X\}_m$. This implies that $A_{ij}=0$ (or, in the inverse notation, $\rho_{ij}=0$) when i and j take on values of the subscripts of the X's in different sets. Thus, in the case of independence, the matrix $\|A_{ij}\|$ will consist of blocks of non-zero elements down the main diagonal with zero elements everywhere else. Let the A's and ρ's which are zero under the hypothesis of independence, be denoted by $[A]$ and $[\rho]$ respectively. We shall refer to the hypothesis of independence as H_I. To test H_I we must devise a criterion for measuring the joint departure of the sample estimates of the A's in $[A]$ (or the ρ's in $[\rho]$) from zero, and interpret its value in terms of probability.

Statistisk Tidskrift, Vol. VIII (1928). Analytical properties of certain general statistical functions of a given sample are discussed by Frisch with special reference to the problem of detecting linear relationships among variables from observations. His discussion, however, does not include sampling problems and significance tests.

II. DERIVATION OF THE CRITERION

Let us suppose that the sample O has been drawn from some normal population whose law of distribution can be represented by (1). The probability of the occurrence of the sample actually observed with values of the X_i falling into the intervals $X_{i\alpha} \pm dX_{i\alpha}$ ($i = 1, 2, \cdots, n$; $\alpha = 1, 2, \cdots, N$) will be (to within infinitesimals of order $dX_{i\alpha}$),

$$(2) \qquad C = \prod_{\alpha=1}^{N} P_\alpha dX$$

where P_α is (1) X_i($i = 1, 2, \cdots, n$) replaced by $X_{i\alpha}$, and dX is the product of all of the differentials $dX_{i\alpha}$.

We shall derive the criterion by the application of the method of maximum likelihood as used by Neyman and Pearson[2] and discuss its validity from analytical and geometrical considerations. Briefly, the procedure is as follows. We must specify:

(a) The class Ω of all normal n-variate populations U from one of which O is assumed to have been drawn.

(b) The subclass ω of Ω to which O must belong if the hypothesis H_I be true.

We find the maximum of C in (2) for variations of the A's and m's under the assumption that U is, (i) a member of Ω, let this be $C(\Omega$ max); and (ii) a member of ω, namely, those populations in which the A's of $[A]$ are zero; call this $C(\omega$ max). The likelihood of H_I is defined as

$$(3) \qquad \lambda_I = \frac{C(\omega \text{ max})}{C(\Omega \text{ max})}.$$

By the usual methods, it is easy to show that $C(\Omega$ max) occurs when

$$(4) \qquad m_i = \overline{X}_i = \frac{1}{N} \sum_{\alpha=1}^{N} X_{i\alpha}$$

$$(5) \qquad A_{ij} = a^{ij}$$

where a^{ij} is the element in the ith row and jth column of the inverse of $\|a_{ij}\|$ where

$$(6) \qquad a_{ij} = \frac{1}{N} \sum_{\alpha=1}^{N} (X_{i\alpha} - \overline{X}_i)(X_{j\alpha} - \overline{X}_j).$$

$C(\omega$ max) occurs when

$$(7) \qquad m_i = \overline{X}_i$$

[2] *Phil. Trans. Roy. Soc.*, Ser. A, Vol. CCXXXI (1933).

$$A_{i_m i_m} = a^{i_m i_m}$$

(8)

$$(i_m, j_m = \bar{n}_m + 1, \bar{n}_m + 2, \cdots, \bar{n}_m + n_m, m = 1, 2, \cdots, k)$$

where $a^{i_m i_m}$ is the element in the i_m-th row and j_m-th column of the inverse of $\|a_{i_m i_m}\|$.

Placing these values in (2) and taking the ratio as defined by (3) we find at once

(9)
$$\lambda_I = \frac{|a_{ij}|}{\prod_{m=1}^{k} |a_{i_m i_m}|}.$$

We note that the determinants in the denominator of (9) are mutually exclusive principal minors of $|a_{ij}|$ such that each diagonal element in $|a_{ij}|$ is contained in one of them. Now, the sample value r_{ij} of the correlation coefficient ρ_{ij} is defined as $r_{ij} = \dfrac{a_{ij}}{\sqrt{a_{ii} a_{jj}}}$. Hence, by dividing the numerator and denominator of (9) by $a_{11} a_{22}, \cdots, a_{nn}$ we get λ_I expressed in terms of sample correlations:

(10)
$$\lambda_I = \frac{|r_{ij}|}{\prod_{m=1}^{k} |r_{i_m i_m}|}.$$

For convenience, we shall denote by $[r]$ the r's corresponding to the ρ's in $[\rho]$. Thus, $[r]$ is the set of r's contained in the numerator of (10) which are not contained in the denominator. It can be readily shown that if the A's involved in determining $C(\Omega \max)$ are expressed in terms of other population parameters, e.g., σ's and ρ's such that the Jacobian of the transformation of parameters does not vanish, the λ_I criterion remains invariant.

III. INTERPRETATION OF λ.

1. *Analytical properties.* In examining H_I we are not primarily concerned with the degree of relationship of variables *within* sets, which is measured by sample values of correlation coefficients or covariances within the groups. The important thing is to study the departure of the correlation of variables between sets from zero.

In testing H_I from the sample, our decision is subject to two kinds of error:[3]

(a) Rejecting H_I when it is true,

(b) Accepting H_I when it is false.

[3] See Neyman and Pearson, *loc. cit.* p. 296.

The decision will be made from the value of λ_I, and the degree of confidence to be placed in the decision will be determined from the frequency function of λ_I when H_I is true. It is therefore important to set up a correspondence between the tenability of H_I and the values which can be taken on by λ_I.

It is well known from the theory of Gramian determinants that determinants of type $|r_{ij}|$ and $|r_{i_m i_m}|$ are positive definite. This also follows from the sampling properties of $|r_{ij}|$ and its principal minors. An important property of λ_I is that it lies between 0 and 1. Perhaps the simplest proof of this fact follows at once from the sampling properties of λ_I. We shall assume the result now and give the proof in Section IV.

We shall now prove the following property of λ_I: If $|r_{ij}|$ and $|r_{i_m i_m}|$ $(m = 1, 2, \cdots, k)$ are not zero, then λ_I can be unity for variations of the r's in $[r]$ and fixed values of the remaining r's if and only if all r's in $[r]$ vanish. It is obvious that λ_I is unity when the r's in $[r]$ vanish. To prove the converse, namely, that if λ_I is unity the r's in $[r]$ vanish, it is sufficient to show that $|r_{ij}|$ has only the maximum $\prod_{m=1}^{k} |r_{i_m i_m}|$, which will occur when the r's in $[r]$ vanish. Let R_{ij} be the cofactor of r_{ij} in $|r_{ij}|$ and $[R]$ the set of R's corresponding to the set $[r]$. For a maximum, it is necessary that the partial derivatives of $|r_{ij}|$ with respect to the r's in $[r]$ vanish. This means that all R's in $[R]$ must be zero. Now, if $|r_{ij}|$ does not vanish, there is a one-to-one correspondence between the r's and R's, that is, each point in the r-space corresponds to one and only one point in the R-space. Expressing $r_{\alpha\beta}$ in terms of the R's we have

$$(11) \qquad r_{\alpha\beta} = \frac{\bar{R}_{\alpha\beta}}{|R_{ij}|^{\frac{n-2}{n-1}}}$$

where $\bar{R}_{\alpha\beta}$ is the cofactor of $R_{\alpha\beta}$ in $|R_{ij}|$. Now, if $r_{\alpha\beta}$ is any r in $[r]$ it is clear from (11) that $r_{\alpha\beta}$ vanishes if all R's in $[R]$ are zero. Since the correspondence is one-to-one when $|r_{ij}| \neq 0$ we therefore have that the only solution of the equations obtained by setting the R's in $[R]$ equal to zero is that all r's in $[r]$ vanish. Thus, the only stationary point of $|r_{ij}|$ for variations of the r's in $[r]$ occurs when these r's are zero. The value of $|r_{ij}|$ at this point is clearly $\prod_{m=1}^{k} |r_{i_m i_m}|$. That the point is actually a maximum follows at once from the continuity of $|r_{ij}|$ and the fact that λ_I cannot exceed unity.

We have shown that λ_I is unity if and only if the r's in $[r]$ vanish. Thus, for given values of observed correlations within sets, H_I is rigorously true as far as the information furnished by the sample is concerned only when all observed correlations between groups are zero.

Considering the lower end of the scale, we find that λ_I vanishes when there is linear independence of variables within groups but linear dependence of variables between at least two groups. This means that λ_I is zero when there is perfect correlation between at least one pair of linear functions l_1 and l_2 (having non-zero coefficients) of the variables $X_1, X_2, \cdots, X_n$, where the variables in l_1 and l_2 are members of at least two of the sets $\{X\}_m$. Therefore, any condition under which λ_I vanishes implies that H_I is completely unsupported by the sample.

From an analytical point of view we have seen that as λ_I varies from zero to unity H_I becomes more and more tenable on basis of the sample O.

2. *Geometrical Interpretation.* First we shall consider only two sets of variables $\{X\}_1$ and $\{X\}_2$ having n_1 and n_2 variables respectively. Let $X_{i\alpha}(i=1, 2, \cdots, n)$ be represented as coordinates of a point in n-dimensional Euclidean space S_n. Then O will be represented by N points in S_n. Denote by S_{n_1} and S_{n_2} the subspaces of S_n having coordinate axes corresponding to the variables in $\{X\}_1$ and $\{X\}_2$. Now consider the N points in S_{n_1} obtained by projecting the points represented by O in S_n orthogonally onto S_{n_1}. Let $X_{i\alpha}' = X_{i\alpha} - \overline{X}_i$. The point P whose coordinates are X_p $(p=1, 2, \cdots, n_1)$ and any other set of n_1 observational points of S_{n_1} determine an n_1-dimensional tetrahedron or simplex whose volume is given by

$$(12) \qquad\qquad V = \frac{1}{n_1!} \left| X_{p\alpha_t}' \right|$$

where $\alpha_t(t=1, 2, \cdots, n_1)$ is a set of n_1 values of $1, 2, \cdots, N$. The size of V will indicate how far the simplex is from being flattened into fewer dimensions than n_1, for V can vanish only when the n_1+1 points lie in fewer dimensions than n_1. Since V can be negative as well as positive it is reasonable to take V^2 as a positive measure of how flat the simplex is. But

$$(13) \qquad V^2 = \frac{1}{(n_1!)^2} \left| \sum_t X_{p\alpha_t}' X_{q\alpha_t}' \right|. \qquad (p, q = 1, 2, \cdots, n_1)$$

Now consider the $\dfrac{N!}{n_1!(N - n_1)!}$ n_1-dimensional simplexes which can be formed from the N projected sample points in S_{n_1} such that P is included in each. The average of the squares of the volumes of these simplexes suggests itself[4] as a measure of how far all of the points are

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[4] Compare Frisch *loc. cit.*

from being in a space of fewer dimensions than n_1. It can be shown[5] that this average, say $\bar{V}_1{}^2$, is

$$(14) \qquad \bar{V}_1{}^2 = G \left| a_{pq} \right|$$

where a_{pq} has been defined in (6), and G is a constant depending only on n_1 and N.

Next we shall consider deviations of each of the variables $X_p{}'(p = 1, 2, \cdots, n_1)$ from its corresponding least-square regression plane on $X_r{}'(r = n_1+1, \cdots, n)$. To find the coefficients in the regression planes we determine the b's so as to minimize each of the sums of squares

$$(15) \qquad \sum_{\alpha=1}^{N}\left(X_{p\alpha}' - \sum_{s=n_1+1}^{n} b_{ps} X_{s\alpha}' \right)^2 \qquad (p = 1, 2, \cdots, n_1).$$

Accordingly, we easily find

$$(16) \qquad b_{ps} = \frac{(a)_{ps}}{a}$$

where a is the determinant $\left| a_{rs} \right| (r, s = n_1+1, \cdots, n)$ and $(a)_{ps} = a \sum_r a_{pr} a^{rs}$. Now let

$$(17) \qquad X''_{p\alpha} = X'_{p\alpha} - \sum_{s=n_1+1}^{n} \frac{(a)_{ps}}{a} X'_{s\alpha}$$

and

$$(18) \qquad c_{pq} = \frac{1}{N} \sum_{\alpha=1}^{N} X''_{p\alpha} X''_{q\alpha}.$$

The deviations $X''_{p\alpha}$ can be interpreted in an n_1-dimensional space in exactly the same way in which the $X'_{p\alpha}$ were interpreted in S_{n_1}. There is a correspondence of simplexes formed by the $X''_{p\alpha}$ and those formed by the $X'_{p\alpha}$. The simplexes formed by the $X''_{p\alpha}$ will tend to be smaller in volume (although some of them may actually be larger) than those formed by the $X'_{p\alpha}$ simply because of the minimizing restrictions imposed on (15). The average of the sum of squares of the volumes of these simplexes is found to be

$$(19) \qquad \bar{V}_2{}^2 = G \left| c_{pq} \right|.$$

It is clear that $\bar{V}_2{}^2$ measures the same thing that $\bar{V}_1{}^2$ does after the variation due to a linear function of the X's in $\{X\}_2$ has been eliminated by least squares. If no evidence of such variation is furnished by the sample then all of the b's in (16) will vanish and $\bar{V}_2{}^2 = \bar{V}_1{}^2$. If this variation is present, then $\bar{V}_2{}^2$ will be less than $\bar{V}_1{}^2$ and the quantity

[5] Kowalewski, *Determinantentheorie*, Veit and Co., Leipzig, pp. 65–72.

$1 - \dfrac{\overline{V}_2{}^2}{\overline{V}_1{}^2}$ offers a measure of the amount of variation, that is, a comprehensive measure of the relationship or dependence between $\{X\}_1$ and $\{X\}_2$. We shall show that $\dfrac{\overline{V}_2{}^2}{\overline{V}_1{}^2}$ is identical with λ_I for the case of of two groups of variables. Expanding c_{pq}, we get

$$(20) \qquad c_{pq} = a_{pq} - \sum_r \frac{(a)_{pr}}{a} a_{pr} - \sum_s \frac{(a)_{ps}}{a} a_{ps} + \sum_{r,s} \frac{(a)_{pr}(a)_{ps}}{a^2} a_{rs}.$$

Since $a_{rs} = a_{sr}$ it can be readily verified that $\sum (a)_{ps} a_{rs} = a a_{pr}$ and hence

$$(21) \qquad \sum_{r,s} \frac{(a)_{pr}(a)_{ps}}{a^2} a_{rs} = \sum_r \frac{(a)_{pr}}{a} a_{pr}.$$

Therefore

$$(22) \qquad c_{pq} = a_{pq} - \sum_s \frac{(a)_{ps}}{a} a_{ps} = \frac{|a_{ij}|_{pq}}{a},$$

where $|a_{ij}|_{pq}$ denotes the determinant obtained by deleting all except the pth row and qth column of the first n_1 rows and columns of $|a_{ij}|$. Now, it can be shown by Sylvester's theorem[6] that

$$(23) \qquad |c_{pq}| = \frac{|a_{ij}|}{|a_{rs}|}$$

and therefore

$$(24) \qquad \frac{\overline{V}_2{}^2}{\overline{V}_1{}^2} = \frac{|a_{ij}|}{|a_{pq}| \cdot |a_{rs}|},$$

which is λ_I for the case of two groups. For convenience, denote this by $\lambda(1, 2)$. Let us return to the case of k sets of variables. λ_I, for the general case, can be expressed in various ways as a product of $k-1$ of such 2-group λ's. It is sufficient to consider only one such representation:

$$(25) \quad \lambda_I = \lambda(1, 2 + 3 + \cdots + k)\lambda(2, 3 + 4 + \cdots + k)\lambda(k - 1, k).$$

The geometrical interpretation of each of these λ's is similar to that of $\lambda(1, 2)$. For example, $1 - \lambda(1, 2+3 + \cdots + k)$ furnishes a measure of the amount of dependence between $\{X\}_1$ and the set of all variables contained in $\{X\}_2, \{X\}_3, \cdots, \{X\}_k$. Clearly, λ_I will be unity when and only when all λ's on the right in (28) are unity, which can only happen when the b's corresponding to those in (16) all vanish for each λ. This, in turn, implies that none of the variation in the sense mentioned

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[6] Kowalewski, *loc. cit.*, p. 86–88.

before in any set of variables can be ascribed to that in any other set or linear combination of sets. λ_I will be zero when one or more of the λ's vanish.

Another somewhat simpler geometrical interpretation of λ_I which supports its validity as a criterion for testing H_I can be arrived at as follows:

Consider an N-dimensional Euclidean space S_N with coordinate axes $X_{0\alpha}(\alpha = 1, 2, \cdots, N)$, each dimension representing an individual of the sample. The N observations of the n variables in the sample O will be represented by n points in S_N. The point P_i will lie in the plane π_i whose equation is $\sum_{\alpha=1}^{N} X_{0\alpha} = N\overline{X}_i$. π_i intersects the straight line $X_{01} = X_{02} = \cdots = X_{0N}$ in a point P_i' whose coordinates are $(\overline{X}_i, \overline{X}_i, \cdots, \overline{X}_i)$. Now let each of the segments P_iP_i' be moved parallel to itself so as to make the points P_i' coincide. For convenience let them coincide at the origin. Then clearly the planes π_i, which are $N-1$ dimensional, also coincide. In their new position the segments P_iP_i' will now form a corner which will determine a parallelotope of n dimensions imbedded in $N-1$ dimensional space. The edges P_iP_i' of the parallelotope clearly have the lengths $\sqrt{a_{ii}}$, where a_{ii} is defined by (6). The cosine of the angle between the i-th and j-th edges is r_{ij} as defined by (10). The volume of the parallelotope is:[7]

$$(26) \qquad V = \left\{ a_{11}a_{22} \cdots a_{nn} \, | \, r_{ij} | \right\}^{\frac{1}{2}}.$$

Now consider the parallelotopes determined by the sets of segments $\{P_{im}P_{im}'\}$ $(m = 1, 2, \cdots, k)$. If V_m is the volume of the mth parallelotope, then

$$(27) \qquad V_{n_m} = \left\{ a_{n_m+1, \bar{n}_m+1} \cdots a_{\bar{n}_m+m, \, \bar{m}+m} \cdot \, | \, r_{i_m j_m} | \right\}^{\frac{1}{2}}.$$

If these parallelotopes are mutually perpendicular, that is, if each pair of the segments P_iP_i' not contained in the same set $\{P_{im}P'_{im}\}$ is orthogonal, then V is identical with the product $V_1V_2 \cdots V_k$. The quantity $\dfrac{V}{V_1V_2 \cdots V_k}$ therefore offers a measure of the degree to which the parallelotopes are perpendicular to each other. But this ratio is $\sqrt{\lambda_I}$.

From the foregoing analytical and geometrical considerations it appears that λ_I is an appropriate criterion for testing H_I. To measure the degree to which H_I is tenable on basis of an observed value of λ_I we must find the sampling distribution of λ_I.

[7] Cf. D. M. Y. Sommerville, *An Introduction to the Geometry of N Dimensions*, Methuen & Co., London, pp. 119–123.

IV. SAMPLING MOMENTS AND DISTRIBUTION OF λ_I

1. *Moments and exact distributions.* We shall find the moments and distribution of λ_I under the assumption that H_I is true, that is, that O is from a normal population in which the ρ's in $[\rho]$ are zero. Under this assumption the h-th moment M_h of λ_I was found elsewhere[3] to be

$$(28) \quad M_h = \prod_{m=1}^{k} \prod_{\mu=1}^{n_m} \left[\frac{\Gamma\left(\dfrac{N-\mu}{2}\right)}{\Gamma\left(\dfrac{N-\mu}{2}+h\right)} \right] \prod_{i=1}^{n} \left[\frac{\Gamma\left(\dfrac{N-i}{2}+h\right)}{\Gamma\left(\dfrac{N-i}{2}\right)} \right].$$

If we let $\lambda_I = \omega$ and use the relation

$$(29) \quad \frac{\Gamma(m+h)}{\Gamma(m+n+h)} = \frac{1}{\Gamma(n)} \int_0^1 \theta^{n+h-1}(1-\theta)^{n-1} d\theta,$$

we find that the k-th moment of ω is identical with that of $\theta_1 \theta_2 \cdots \theta_{m-n_1}$ where the θ's are distributed according to the law

$$f(\theta_1, \theta_2, \cdots, \theta_{n-n_1})$$

$$(30)$$
$$= K \prod_{m=2}^{k} \prod_{\mu=1}^{n_m} \left[\theta_{\bar{n}_m - n_1 + \mu}^{\frac{N - \bar{n}_m - \mu}{2} - 1} (1 - \theta_{n_m - n_1 + \mu})^{n_m/2 - 1} \right]$$

where each θ lies on the interval $(0, 1)$, and

$$(31) \quad K = \prod_{m=2}^{k} \prod_{\mu=1}^{n_m} \left[\frac{\Gamma\left(\dfrac{N-\mu}{2}\right)}{\Gamma\left(\dfrac{\bar{n}_m}{2}\right) \Gamma\left(\dfrac{N - \bar{n}_m - \mu}{2}\right)} \right].$$

As before, $\bar{n}_m = n_1 + n_2 + \cdots n_{m-1}$. Now it follows from the uniqueness of the solution of the moment problem for a finite interval that the distribution of ω is identical with that of the product $\theta_1 \theta_2 \cdots \theta_{n-n_1}$. From this it is obvious that ω and hence λ_I has a maximum of unity, a fact which was assumed in Section III.

To find the distribution of ω it is sufficient to substitute

$$\theta_1 = \frac{\omega}{\theta_2 \theta_3 \cdots \theta_{n-n_1}} \text{ in (30) and integrate with respect to } \theta_2, \theta_3, \cdots, \theta_{n-n_1}$$

The general result is rather complicated and will not be given here. It

[3] Wilks, *Biometrika*, XXIV (1932), 491–494.

is a special case of a general distribution considered by the author[9] elsewhere. However, we shall consider exact distributions of ω for special sets of values of k and the n_m.

It is perhaps of more practical value to obtain the integrals than the distributions themselves. Thus, if $df(\omega)$ is the probability that ω will fall in the range $\omega \pm d\omega$, then in repeated samples from (1) the probability of obtaining a value of ω less than an observed value, say W, is $\int_0^W df(\omega)$. Denote this integral by $J_W(n_1, n_2, \cdots, n_k; N)$ which has the same value for all permutations of the n's because of the symmetry of λ_I with respect to the sets of variables $\{X\}_m$. It will be noted, of course, that the distribution of ω can be found by differentiating $J_\omega(n_1, n_2, \cdots, n_k; N)$ with respect to W, afterwards replacing W by ω. In finding J_W for the various cases, the moments were simplified as much as possible by the formula

$$(32) \qquad \Gamma(\alpha + \tfrac{1}{2})\Gamma(\alpha + 1) = \frac{\sqrt{\pi}\,\Gamma(2\alpha + 1)}{2^{2\alpha}}.$$

We also use the notation

$$B_W(p, q) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} \int_0^W x^{p-1}(1 - x)^{q-1}dx$$

which can be evaluated from the tables of the Incomplete Beta Function.[10] The integrals for the simplest cases of ω are:

$$(33) \qquad J_W(1, n - 1; N) = B_W\left(\frac{N - n}{2}, \frac{n - 1}{2}\right).$$

In this case λ_I reduces to $1 - R^2$ where R is the multiple correlation coefficient between one of the variables and the remaining $n-1$ variables.

$$(34) \quad J_W(2, n-2; N) = B_{\sqrt{W}}(N-n, n-2)$$

$$(35) \qquad J_W(3, 3; N) = B_W\left(\frac{N-4}{2}, \frac{1}{2}\right)$$

$$+ \frac{\Gamma(N-2)\Gamma\left(\dfrac{N-3}{2}\right)}{6\sqrt{\pi}\,\Gamma(N-6)\Gamma\left(\dfrac{N-4}{2}\right)} W^{(N-6)/2}\left\{ 2\sqrt{1-W}\left(\frac{1}{N-6} - \frac{W}{N-3}\right)\right.$$

$$\left. - \frac{3}{N-5} W^{1/2}(\pi - 2\sin^{-1}\sqrt{W}) + \frac{6W}{N-4}\log_e\left(\frac{1+\sqrt{1-W}}{\sqrt{W}}\right)\right\}.$$

[9] *Loc. cit.*, p. 475.

[10] Edited by K. Pearson, Issued (1934) by the Biometric Laboratory, University of London.

$$(36) \quad J_W(3, 4; N) = W^{(N-5)/2} + \frac{\Gamma(N-2)(N-5)}{48\Gamma(N-7)} W^{(N-7)/2} \left\{ \frac{1}{N-7} - \frac{8}{N-6}\sqrt{W} \right.$$

$$\left. + W\left(-\frac{1}{N-7} + \frac{8}{N-6} + \frac{1}{N-3} - \frac{6}{N-5}\log_e W \right) - \frac{1}{N-3}W^{3/2} \right\}.$$

$$(37) \quad J_W(4, 4; N) = B_{\sqrt{W}}(N-6, 4) + \frac{\Gamma(N-2)\Gamma(N-4)}{36\Gamma(N-6)\Gamma(N-8)} W^{(N-8)/2} \left\{ \frac{1}{20(N-8)} \right.$$

$$- \frac{3W^{1/2}}{4(N-7)}(1 - \sqrt{W})^4 + W\left[-\frac{11}{2(N-6)} - \frac{1}{2(N-8)} - \frac{1}{N-5} \right.$$

$$\left. - \frac{3\log_e W}{2(N-6)} \right] + W^{3/2}\left[-\frac{3}{2(N-5)} + \frac{1}{N-8} + \frac{9}{N-6} - \frac{3}{2(N-5)}\log_e W \right]$$

$$+ W^2\left[-\frac{3}{4(N-8)} - \frac{9}{2(N-6)} + \frac{3}{N-5} \right]$$

$$\left. + W^{5/2}\left[\frac{1}{5(N-8)} + \frac{1}{N-6} - \frac{1}{2(N-5)} \right] \right\}.$$

$$(38) \quad J_W(1, 1, n-2; N) = B_W\left(\frac{N-2}{2}, \frac{1}{2} \right) + \frac{\Gamma(N-2)\Gamma\left(\frac{N-1}{2} \right)}{\Gamma(N-n)\Gamma\left(\frac{N-2}{2} \right)\Gamma(n-2)}$$

$$\times \sum_{i=0}^{n-1} \frac{(-1)^i \binom{n-3}{i} W^{(N-n+i)/2}\Gamma\left(\frac{N-2-i}{2} \right)}{(N-n+i)\Gamma\left(\frac{n-1-i}{2} \right)} \left[1 - B_W\left(\frac{n-2-i}{2}, \frac{1}{2} \right) \right].$$

$$(39) \quad J_W(1, 2, 2; N) = B_W\left(\frac{N-5}{2}, 2 \right) + \frac{\Gamma(N-2)\Gamma\left(\frac{N-1}{2} \right)}{\Gamma(N-4)\Gamma\left(\frac{N-5}{2} \right)} W^{(N-5)/2}\left[\frac{2}{N-4} \right.$$

$$\left. - \frac{1}{N-3} - \frac{4\sqrt{W}}{N-4} + W\left(\frac{2}{N-4} + \frac{1-\log_e W}{N-3} \right) \right].$$

$$(40) \quad J_W(1, 2, 3; N) = B_W\left(\frac{N-6}{2}, \frac{5}{2} \right)$$

$$+ \frac{2\Gamma(N-2)\Gamma\left(\frac{N-1}{2} \right)}{3\sqrt{\pi}\Gamma(N-5)\Gamma\left(\frac{N-6}{2} \right)} W^{(N-6)/2} \left\{ \sqrt{W}\left(\frac{W}{N-3} - \frac{2}{2(N-5)} \right)\pi \right.$$

$$+ \left(\frac{3}{N-5} - \frac{2W}{N-3} \right)\sin^{-1}\sqrt{W} + \sqrt{1-W}\left[\frac{2}{N-5} - \frac{2}{N-4} + \frac{2}{3(N-3)} \right.$$

$$\left. + W\left(\frac{1}{N-5} - \frac{4}{N-4} - \frac{8}{3(N-3)} \right) \right] + \frac{6}{N-4} W\log_e\left(\frac{1+\sqrt{1-W}}{\sqrt{W}} \right) \right\}.$$

(41) $J_W(1, 2, 4; N) = B_W\left(\dfrac{N-7}{2}, 3\right) + \dfrac{\Gamma(N-2)\Gamma\left(\dfrac{N-1}{2}\right)}{12\Gamma(N-6)\Gamma\left(\dfrac{N-7}{2}\right)} W^{(N-7)/2}\left\{\left[\dfrac{2}{N-6}\right.\right.$

$$-\frac{3}{N-5} + \frac{2}{N-4} - \frac{1}{2(N-3)}\right] - \frac{16\sqrt{W}}{3(N-6)} + W\left[\frac{4}{N-6} - \frac{12}{N-4}\right.$$

$$\left. + \frac{2}{N-3} - \frac{6}{N-5}\log_e W\right] + \frac{16W^{3/2}}{N-4}$$

$$+ W^2\left[-\frac{3}{2(N-3)} - \frac{2}{3(N-6)} + \frac{3}{N-5} - \frac{6}{N-4} + \frac{1}{N-3}\log_e W\right]\right\}.$$

(42) $J_W(2, 2, 2; N) = B_{\sqrt{W}}(N-6, 4) + \dfrac{\Gamma^2(N-2)}{6\Gamma(N-4)\Gamma(N-6)} W^{(N-)6/2}\left\{\dfrac{1}{2(N-4)}\right.$

$$-\frac{1}{3(N-3)} + \sqrt{W}\left[\frac{3}{2(N-3)} - \frac{3}{N-4}\right] + W\left[\frac{3}{2(N-4)}(1-\log_e W)\right.$$

$$\left. - \frac{3}{N-3}\right] + W^{3/2}\left[\frac{1}{N-4} + \frac{11}{6(N-3)} - \frac{1}{2(N-3)}\log_e W\right]\right\}.$$

(43) $J_W(2, 2, 3; N) = B_{\sqrt{W}}(N-7, 5) + \dfrac{\Gamma^2(N-2)}{48\Gamma(N-5)\Gamma(N-7)} W^{(N-7)/2}\left\{\dfrac{1}{2(N-5)}\right.$

$$-\frac{2}{3(N-4)} + \frac{1}{4(N-3)} + \sqrt{W}\left[\frac{4}{N-4} - \frac{4}{N-5} - \frac{4}{3(N-3)}\right]$$

$$+ W\left[\frac{3}{N-3} - \frac{12}{N-4} - \frac{3}{N-5}\log_e W\right]$$

$$+ W^{3/2}\left[\frac{4}{N-5} + \frac{20}{3(N-4)} - \frac{4}{N-4}\log_e W - \frac{4}{N-3}\right]$$

$$+ W^2\left[\frac{25}{12(N-3)} - \frac{1}{2(N-5)} + \frac{2}{N-4} - \frac{1}{2(N-3)}\log_e W\right]\right\}.$$

For general values of N the integrals of the distribution functions of the ω's of higher order than those just considered are not so readily expressible in terms of known functions. However, they can always be expressed in certain generalized hypergeometric series.

2. *Approximate distribution for large samples.* If N is large compared with n we shall prove that the quantity $N(1-\omega)$ is approximately distributed according to the function

(44) $$dF(\chi^2) = \frac{\left(\dfrac{1}{2}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)} e^{-\chi^2/2}(\chi^2)^{\nu/2-1}d(\chi^2),$$

where

$$\nu = \frac{n(n+1)}{2} - \sum_{m=1}^{k}\frac{n_m(n_m+1)}{2}.$$

That is, for large samples $N(1-\omega)$ has approximately the χ^2 distribution with ν degrees of freedom. Therefore, the value of $J_W(n_1, n_2, \cdots, n_k; N)$ is approximated by

$$(45) \qquad \int_{N(1-W)}^{\infty} dF(x^2) = 1 - I\left(\frac{N(1-W)}{\sqrt{2\nu}}, \frac{\nu-2}{2}\right),$$

where

$$(46) \qquad I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} e^{-x}x^p dx,$$

the Incomplete Gamma function,[11] which has been tabled for values of p up to 50. Values of w for which (45) has various values from .01 to .09 can be found from Fisher's[12] tables.

To prove that (44) is the limiting form of the distribution of $N(1-\omega)$ it is sufficient to consider the distribution of a variable θ whose hth moment is

$$(47) \quad M_h(\theta) = K \frac{\Gamma\left(\frac{N}{2}+b_1+h\right)\Gamma\left(\frac{N}{2}+b_2+h\right) \cdots \Gamma\left(\frac{N}{2}+b_r+h\right)}{\Gamma\left(\frac{N}{2}+c_1+h\right)\Gamma\left(\frac{N}{2}+c_2+h\right) \cdots \Gamma\left(\frac{N}{2}+c_r+h\right)}$$

where K is such that $M_0(\theta)=1$ and $-N/2<b_i<c_i(i=1, 2, \cdots, r)$, the b's and c's being independent of N. The distribution of θ can be written[13]

$$(48) \quad \begin{aligned} dg(\theta) &= K'\theta^{N/2+b_r-1}(1-\theta)^{\gamma_r-\beta_r-1}d\theta \\ &\int_0^1 \int_0^1 \cdots \int_0^1 \nu_1^{\gamma_r-1-\beta_r-1}\nu_2^{\gamma_r-1-\beta_r-1-1} \cdots \nu_{r-1}^{\gamma_1-\beta_1-2} \\ &\times (1-\nu_1)^{c_1-b_1-1}(1-\nu_2)^{c_2-b_2-1} \cdots (1-\nu_{r-1})^{c_{r-1}-b_{r-1}-1} \\ &\times [1-(1-\nu_1)(1-\theta)]^{b_1-c_2} \\ &\times [1-(1-\nu_1\nu_2)(1-\theta)]^{b_2-c_3} \cdots \\ &\times [1-(1-\nu_1\nu_2 \cdots \nu_{r-1})(1-\theta)]^{b_{r-1}-c_r}d\nu_1 \cdots d\nu_{r-1} \end{aligned}$$

where

$$\gamma_i = \sum_{j=0}^{i-1} c_{r-j}, \qquad \beta_i = \sum_{j=0}^{i-1} b_{r-j} \qquad (i = 1, 2, \cdots, r)$$

and

[11] Edited by K. Pearson, Biometric Laboratory (1922), Univ. of London.

[12] R. A. Fisher, *Statistical Methods for Research Workers*, 4th edition (1932), Oliver & Boyd, Edinburgh, p. 150.

[13] Wilks, *loc. cit.*, p. 475.

$$K' = \prod_{i=1}^{r} \left[\frac{\Gamma\left(\dfrac{N}{2} + c_i\right)}{\Gamma\left(\dfrac{N}{2} + b_i\right)\Gamma(c_i - b_i)} \right].$$

If we let $\theta = 1 - \chi^2/N$ in (48) and apply the formula

(49) $$\Gamma(x) = e^{-x}x^{x-1/2}\sqrt{2\pi}\left(1 + \frac{1}{12x} + \cdots\right)$$

to all gamma functions in K' involving N, we find that

(50)
$$\begin{aligned}
&\lim_{N\to\infty} \left[K'\left(1 - \frac{\chi^2}{N}\right)^{N/2+b_r-1}\left(\frac{\chi^2}{N}\right)^{\gamma_r-\beta_r-1}\frac{d(\chi^2)}{N}\right] \\
&= \frac{e^{-\chi^2/2}\left(\dfrac{\chi^2}{2}\right)^{\gamma_r-\beta_r-1}d(\chi^2)}{2\displaystyle\prod_{i=1}^{r}\Gamma(c_i - b_i)}.
\end{aligned}$$

The function of ν's under the integral in (48) is positive and, after the substitution $\theta = 1 - \chi^2/N$ has been made, the resulting function is integrable for all values of $N (> \chi^2$, of course). It is dominated by the integrable function obtained by replacing

$$\left[1 - (1 - \nu_1\nu_2\cdots\nu_i)\left(\frac{\chi^2}{N}\right)\right]^{b_i-c_i-1} \quad (i = 1, 2, \cdots, r-1)$$

$$\text{by 1 or } \left(1 - \frac{\chi^2}{N}\right)^{b_i-c_i-1}$$

according to whether $b_i - c_{i+1}$ is $\geqq 0$ or < 0. Hence, the limit of the integral is equal to the integral of the limit of the integrand as $N \to \infty$. But the integral of the limit of the integrand is

(51) $$\frac{\displaystyle\prod_{i=1}^{r}\Gamma(c_i - b_i)}{\Gamma(\gamma_r - \beta_r)}.$$

Therefore

(52) $$\lim_{N\to\infty} dg\left(1 - \frac{\chi^2}{N}\right) = \frac{e^{-\chi^2/2}\left(\dfrac{\chi^2}{2}\right)^{\gamma_r-\beta_r-1}}{2\Gamma(\gamma_r - \beta_r)}d(\chi^2).$$

Now the h-th moment of ω as given by (28) is a special case of $M_h(\theta)$.

In the case of ω, $r = n$ and $\gamma_n - \beta_n = \nu/2$, which completes the proof that the limiting form of the distribution of $N(1-\omega)$ is given by (44).

V. APPLICATIONS

1. *Deviations from regression functions.* Thus far we have considered only deviations of the observations of each of the n variables X_i from their respective means $\overline{X}_i$. The a_{ij} and r_{ij} have been functions of these deviations. In such types of statistical material as time series it is important to consider deviations of observations from certain regression functions of time. In the study of yields of cereals from plots of varying fertility, the effect of the non-uniform fertility must be eliminated by considering deviations of yields from appropriate regression functions of plots. In the case of time series it may be desirable in a particular problem to eliminate the trend of X_i by means of a second degree polynomial in t, say $a + bt + ct^2$ where a, b, and c, are determined by minimizing the sums of squares

$$\sum_{\alpha=1}^{N} (X_{i\alpha} - a_i - b_i t_\alpha - c_i t_\alpha^2)^2 \quad (i = 1, 2, \cdots, n).$$

If the residuals $(X_i - a_i - b_i t - c_i t^2)$ are distributed like the X_i in (1) then all of the sampling theory developed concerning λ_I holds with N replaced by $N-2$. That is, two additional degrees of freedom are lost in determining b_i and c_i from the data. In general, if $\phi_1, \phi_2, \cdots, \phi_r$ are linearly independent functions of any number of variables $y_1, y_2, \cdots, y_s$ such that $(X_i - a_{i1}\phi_1 - \cdots - a_{ir}\phi_r)$ are distributed like the X_i in (1) and if the a's are determined so as to minimize $\sum_\alpha (X_{i\alpha} - a_{i1}\phi_{1\alpha} - \cdots - a_{ir}\phi_{r\alpha})^2$ where $\phi_{j\alpha} = \phi_j(y_{1\alpha}, y_{2\alpha}, \cdots, y_{s\alpha})$ then the λ_I function of the deviations so obtained has all of the sampling properties of the same function of the deviations $(X_{i\alpha} - \overline{X}_i)$ with N replaced by $N-r+1$. In particular, the probability integral for making a test of significance of an observed value w of λ_I for the independence of k groups of such deviations is $J_n(n_1, n_2, \cdots, n_k; N-r+1)$. The sampling theory of the λ_I's which can be formed within each of the various systems of deviations from regression considered by Bartlett[14] follows at once from those of λ_I by taking account of the number of degrees of freedom in each system. H_I will, of course, refer to the hypothesis of independence of sets of deviations from regression.

2. *Example.* The following correlations have been taken from a matrix of correlations reported by Kelley[15] on 109 seventh-grade pupils:

[14] M. S. Bartlett, *Proc. Roy. Soc. Edin.*, LIII, (1932–33). 260–283.

[15] T. L. Kelley, *Crossroads in the Mind of Man*, Stanford Univ. Press, 1928. p. 114. We have selected the raw correlations for this example because (1) it is desired to study relationships among the quantities actually observed, (2) very

	1	2	3	4	5
1	1	.4249	− .0552	− .0031	.1927
2	.4249	1	− .0416	.0495	.0687
3	− .0552	− .0416	1	.7474	.1691
4	− .0031	.0495	.7474	1	.2653
5	.1927	.0687	.1691	.2653	1

The variables are designed to measure the following five traits respectively:

1. Arithmetic Speed.   4. Social Interest.
2. Arithmetic Power.   5. Activity Interest.
3. Intellectual Interest.

For convenience let the variables 1 and 2 be called set A and 3, 4, and 5, set B. It will be noticed that each of the correlations between A and B are all fairly small with the exception of r_{15}, which is .1927. Furthermore, the correlations within each set are somewhat larger. Therefore the question arises: Is there any significant relationship between the sets A and B? Can they be regarded as having arisen from a universe in which the inter-correlations between A and B are zero. These questions cannot always be adequately answered by testing the significance of the deviations from zero of the correlations between A and B individually, because some may show significance and others may not. We must take into account all of the correlations simultaneously. This is done by means of the λ_I criterion, which, in this case, is the ratio of the determinant of all correlations to the product of the determinants of correlations in A and B. We find $\lambda_I = W = .9422$ and from expression (34) $J_W(2, 3; 109) = .396$ which is the probability[15] in repeated samples of 109 of obtaining a value of λ_I more unfavorable to the hypothesis of independence of A and B than that actually observed. This is clearly a non-significant discrepancy if .05 is adopted as the probability level of significance. If J_W had been as much as .95 we could have been fairly certain of no relationship between A and B so that in fitting a factor pattern to these five variables the simplest pattern would be one with no factors overlapping A and B. However, if A and B are subsets of a larger set of variables as actually considered by Kelley, the problem of selecting the simplest pattern is very

little is known about the sampling theory of correlations corrected for attenuation and hence any inferences which may be drawn from a *set* of such correlations is likely to be misleading. Indeed, the determinant of a matrix of correlations corrected for attenuation can actually be negative—a thing which can never occur with raw correlations. Therefore, until more rigorous sampling theory is provided than that available at present, such quantities as multiple and partial correlations and regression coefficients should be used with caution, especially if the reliability coefficients are not very high.

[15] Using (45) we find the χ^2 approximation of J_W (2, 3; 109) to be .397.

complicated. In this case, even though A and B may be independent, a complete factor pattern may contain factors overlapping A and B unless they can be shown to be independent of the remaining variables.

3. *Independence of subtests.* In designing a test from a battery of subtests it is desirable to have the subtests correlate as highly as possible with some objective criterion which has a correspondence with the trait to be measured by the test, and at the same time to have the subtests as independent of each other as possible. For given variances the more highly independent the subtests, the more information they will furnish when considered jointly. Now if each subtest is taken as a variable the determinant of the intercorrelations of the subtests furnishes a measure of their independence. This determinant is λ_I with each set having one variable.

4. *Problems arising when H_I is not true.* Finally, it is conceivable that in many problems it would be desirable to establish fiducial limits of the population value of λ_I for an observed value of λ_I. This means that we must find at least an approximate sampling distribution for λ_I when the ρ's in $[\rho]$ are different from zero. This problem remains to be solved.

VI. SUMMARY

We have considered a criterion, λ_I, for testing the mutual independence of k sets of normally distributed variables. The criterion is derived as a Neyman-Pearson λ ratio by applying the principle of maximum likelihood and is a function of observations only. It is expressible as the ratio of the determinant of correlation coefficients of all variables to the product of the determinants of the correlation coefficients within the k sets.

Analytical and geometrical properties of λ_I are discussed and shown to support its validity as a criterion for measuring the independence of the k sets of variables.

The sampling distribution of λ_I has been found under the assumption that the k sets of variables are independent in the population and exact expressions for the probability integrals have been given for several of the simple cases. This makes it possible to use λ_I as a criterion for making significance tests of independence. In the case of large samples an approximation has been found for the probability integral which is expressible in terms of the Incomplete Gamma Function.

The sampling theory of λ_I holds for deviations of variables from regression functions, provided the degrees of freedom are properly taken into account.

The numerical application of the theory is illustrated by an example in psychological measurements involving two groups of variables.

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A MACRODYNAMIC THEORY OF
BUSINESS CYCLES[1]

By M. Kalecki

I

In the following all our considerations concern an economic system *isolated and free of secular trend*. Moreover, we make with respect to that system the following assumptions.

1. We call real gross profit B the total real income of capitalists (business men and private capitalists), amortization included, per unit of time. That income consists of two parts, that consumed and that accumulated:

$$(1) \qquad B = C + A.$$

Thus, C is the total volume of consumers' goods consumed by capitalists, while A—*if we disregard savings of workpeople, and their "capitalistic" incomes*—covers goods of all kind serving the purpose of reproduction and expansion of fixed capital, as well as increment of stocks. We shall call A "gross accumulation."

The personal consumption of capitalists, C, is not very elastic. *We assume that C is composed of a constant part, C_1, and a variable part proportionate to the real gross profit λB:*

$$(2) \qquad C = C_1 + \lambda B$$

where λ is a small constant fraction.

From equations (1) and (2) we get:

$$B = C_1 + \lambda B + A$$

and

$$(3) \qquad B = \frac{C_1 + A}{1 - \lambda},$$

i. e., the real gross profit B is proportionate to the sum $C_1 + A$ of the constant part of the consumption of capitalists C_1 and of the gross accumulation A.

---

[1] The term "macrodynamic" was first applied by Professor Frisch in his work "Propagation problems and impulse problems in dynamics" (*Economic Essays in Honour of Gustav Cassel*, London, 1933), to determine processes connected with the functioning of the economic system as a whole, disregarding the details of disproportionate development of special parts of that system.

The gross accumulation A is equal to the sum of the production of capital goods and of the increment of stocks of all kinds.[2] *We assume that the total volume of stocks remains constant all through the cycle.* This is justified in so far as in existing economic systems totally or approximately isolated (the world, U.S.A.) the total volume of stocks does not show any distinct cyclical variations. Indeed, while business is falling off, stocks of finished goods decrease, but those of raw materials and semi-manufactures rise; during recovery there is a reversal of tendencies. From the above we may conclude that *in our economic system the gross accumulation A is equal to the production of capital goods.*

2. We assume further that *the "gestation period" of any investment is θ.* Of course, this by no means corresponds to the reality; θ is merely the average of various actual durations of "gestation periods," and our system in which θ is a constant value is to be considered as a simplified model of reality.

Whenever an investment is made, three stages can be discerned: (1) investment orders, i.e., all the orders for capital goods to serve the purpose of reproduction or expansion of industrial equipment; the total volume of such orders allocated per unit of time will be called I; (2) production of capital goods; the volume of that production per unit of time, equal, as said above, to the gross accumulation, is called A; (3) deliveries of finished industrial equipment; the volume of such deliveries per unit of time will be called L.[3]

The relation of L and I is simple. Deliveries L at the time t are equal to investment orders I at the time $t - \theta$:

$$(4) \qquad\qquad L(t) = I(t - \theta).$$

($I(t)$ and $L(t)$ are investment orders and deliveries of industrial equipment at the time t.)

The interrelationship of A and I is more complicated.

Let us call W the total volume of unfilled investment orders at the moment t. As each investment needs the time θ to be filled, $1/\theta$ of its volume must be executed in a unit of time. Thus, the production of capital goods must be equal to $1/\theta \cdot W$:

$$(5) \qquad\qquad A = \frac{W}{\theta}.$$

---

[2] Industrial equipment in course of construction is not included in "stocks of all kinds"; thus, change in the volume of the industrial equipment in course of construction is involved in the "production of capital goods."

[3] While A is the production of *all* capital goods, L is only that of *finished* capital goods. Thus, the difference $A - L$ represents the volume of industrial equipment in course of construction, per unit of time.

As regards W, it is equal to the total of orders allocated during the period $(t-\theta, t)$. Indeed, since the "gestation period" of any investment is θ, no order allocated during the period $(t-\theta, t)$ is yet finished at the time t, while all the orders allocated before that period are filled. We thus obtain the equation:

$$(6) \qquad W(t) = \int_{t-\theta}^{t} I(\tau)d\tau.$$

According to equations (4) and (5) we get:

$$(7) \qquad A(t) = \frac{1}{\theta} \int_{t-\theta}^{t} I(\tau)d\tau.$$

($A(t)$ is the production of capital goods at the time t.)

Thus A at the time t is equal to the average of investment orders $I(t)$ allocated during the period $(t-\theta, t)$.

3. Let us call K the volume of the existing industrial equipment. The increment of that volume within the given period is equal to the difference between the volume of deliveries of finished equipment and that of equipment coming out of use. If we denote by $K'(t)$ the derivative of K with respect to time, by $L(t)$ the volume of deliveries of industrial equipment per unit of time (as above), and by U the demand for restoration of equipment used up per unit of time, we get:

$$(8) \qquad K'(t) = L(t) - U.$$

We can assume that the demand for restoration of the industrial equipment—U—remains constant all through the cycle. The volume of the existing industrial equipment K shows, it is true, certain fluctuations,
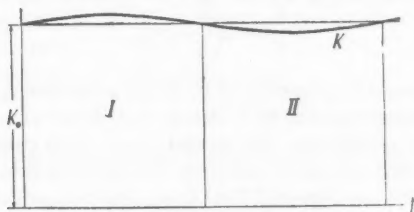
FIGURE 1

e.g., in the first part of the cycle K is above the average, and one might think that then the demand for restoration of equipment ought to be above the average too. Yet, it should be borne in mind that the new equipment is "young" and that its "rate of mortality" is very low, as the average "lifetime" of industrial equipment is much longer than the

duration of a cycle (15–30 years as against 8–12 years). Thus, the fluctuations of the demand for restoration of equipment are of no importance, and may safely be disregarded.

4. The proportions of the investment activity at any time depend on the expected net yield. When the business man will invest a capital k in the construction of industrial equipment, he will first evaluate the probable gross profit b, while deducting (1) the amortization of the capital k, i.e., βk (β—the rate of amortization); (2) the interest on the capital k, i.e., pk (p—the interest rate); (3) the interest on the future working capital, the ratio of which to the invested capital k will be denoted by $\gamma - p\gamma k$. The probable yield of the investment will thus be:

$$\frac{b - \beta k - pk - p\gamma k}{k} = \frac{b}{k} - \beta - p(1 + \gamma).$$

The coefficients β and γ may be considered constant all through the cycle. p is the money rate at the given moment, b/k is the probable future yield evaluated after that of the existing enterprises. The volume of the existing industrial equipment is K, the total real gross profit is B. Thus, the average real gross profit per unit of the existing fixed capital is B/K (that quotient will be called further gross yield B/K).

We may conclude that $\dfrac{b}{k}$ is evaluated after B/K, and that investment activity is controlled by the gross yield B/K and the money rate p. As a matter of fact, the function of B/K and p is not the very volume of investment orders I, but the ratio of that volume to that of industrial equipment K, i.e., I/K. In fact, when B and K rise in the same proportion, B/K remains unchanged, while I rises (probably) as did B and K. Thus, we arrive at the equation:

$$(9) \qquad\qquad \frac{I}{K} = f\left(\frac{B}{K}, p\right)$$

where f is an increasing function of B/K and a decreasing function of p.

It is commonly known that, except for *financial panic* (the so-called crises of confidence), the market money rate rises and falls according to general business conditions. We make on that basis the following simplified assumption: *The money rate p is an increasing function of the gross yield B/K.*

From the assumption concerning the dependence of the money rate p on the gross yield B/K, and from (8), it follows *that I/K is a function of B/K*. As B is proportionate to $C_1 + A$, where C_1, is the constant part of the consumption of capitalists, and A the gross accumulation equal to the production of capital goods, we thus obtain:

$$(10) \qquad \frac{I}{K} = \phi\left(\frac{C_1 + A}{K}\right)$$

ϕ being, of course, an increasing function. We further assume *that ϕ is a linear function*, i.e., that:

$$\frac{I}{K} = m\frac{C_1 + A}{K} - n$$

where the *constant m is positive*, ϕ being an increasing function. Multiplying both sides of the equation by K we get:

$$(11) \qquad I = m(C_1 + A) - nK.$$

$$* \qquad * \qquad *$$

We have seen that between I (investment orders), A (gross accumulation equal to the production of capital goods), L (deliveries of industrial equipment), K (volume of the existing industrial equipment), and the time t, there are interrelationships:

$$(4) \qquad L(t) = I(t - \theta)$$

$$(7) \qquad A(t) = \frac{1}{\theta}\int_{t-\theta}^{t} I(\tau)d\tau$$

$$(8) \qquad K'(t) = L(t) - U$$

resulting from technics of the capitalistic production, and the relation:

$$(11) \qquad I = m(C_1 + A) - nK$$

resulting from the interdependence between investments and yield of existing enterprises. From these equations the relation of I and t may be easily determined.

Let us differentiate (11) with respect to t:

$$(12) \qquad I'(t) = mA'(t) - nK'(t).$$

Differentiating the equation (7) with respect to t, we get:

$$(13) \qquad A'(t) = \frac{I(t) - I(t - \theta)}{\theta}$$

and from (4) and (8):

$$(14) \qquad K'(t) = I(t - \theta) - U.$$

Putting into (12) values of $A'(t)$ and $K'(t)$ from (13) and (14), we have:

(15) $I'(t) = \dfrac{m}{\theta} \left[I(t) - I(t - \theta) \right] - n[I(t - \theta) - U].$

Denoting the deviation of $I(t)$ from the constant demand for restoration of the industrial equipment U by $J(t)$:

(16) $J(t) = I(t) - U,$

we can transform (15) as follows:

$$J'(t) = \frac{m}{\theta} \left[J(t) - J(t - \theta) \right] - nJ(t - \theta)$$

or

(17) $(m + \theta n)J(t - \theta) = mJ(t) - \theta J'(t).$

The solution of that equation will enable us to express $J(t)$ as a function of t and to find out which, if any, are the endogenous cyclical fluctuations in our economic system.

II

It may be easily seen that the equation (17) is satisfied by the function $De^{\alpha t}$ where D is an arbitrary constant value and α a definite value which has to be determined. Replacing $J(t)$ by $De^{\alpha t}$, we get:

$$D(m + \theta n)e^{\alpha(t-\theta)} = Dme^{\alpha t} - D\alpha\theta e^{\alpha t}$$

and, dividing by $De^{\alpha t}$, we obtain an equation from which α can be determined:

(18) $(m + \theta n)e^{-\alpha\theta} = m - \alpha\theta.$

By simple transformations we get further:

$$e^{-m}(m + \theta n)e^{m-\alpha\theta} = m - \alpha\theta$$

and setting

(19) $m - \alpha\theta = z$

(20) $e^{-m}(m + \theta n) = \iota$

we have

(21) $\iota e^z = z$

where z is to be considered as a complex number:

(22) $z = x + iy.$

Thus, (19) can be given the following form:

$$(23) \qquad \alpha = \frac{m - x}{\theta} - i\frac{y}{\theta}$$

and (21) be transformed into:

$$(24) \qquad x + iy = le^z(\cos y + i \sin y).$$

Adopting the method of Tinbergen,[4] we discern two cases: Case I—when $l > 1/e$, and Case II—when $l \leq 1/e$.

Case I. As Tinbergen has shown, in that case all the solutions will be complex numbers, and they will be infinite in number. Let us arrange them by increasing y_k:

$$\cdots x_k - iy_k, \cdots x_2 - iy_2, x_1 - iy_1, x_1 + iy_1, x_2 + iy_2 \cdots x_k + iy_k \cdots.$$

(It is easy to see that when $x_k + iy_k$ is a root of (24), that equation is satisfied as well by $x_k - iy_k$).

From the equation (23) we get values of α:

$$\alpha_k = \frac{m - x_k}{\theta} - i\frac{y_k}{\theta}$$

and

$$\alpha_{-k} = \frac{m - x_k}{\theta} + i\frac{y_k}{\theta}.$$

Functions:

$$D_k e^{\alpha_k t} = D_k e^{(m - x_k)t/\theta}\left(\cos y_k \frac{t}{\theta} - i \sin y_k \frac{t}{\theta}\right)$$

and

$$D_{-k} e^{\alpha_{-k} t} = D_{-k} e^{(m - x_k)t/\theta}\left(\cos y_k \frac{t}{\theta} + i \sin y_k \frac{t}{\theta}\right)$$

satisfy (17).

The general solution of (17), which is at the same time a differential and a functional equation, depends upon the form of the function $J(t)$ in the initial interval $(0, \theta)$; that form is quite arbitrary. Yet, we can develop (with sufficient approximation) the function $J(t)$ in the initial interval into the series $\sum D_k e^{\alpha_k t}$ where the constants D_k depend upon the form of the function $J(t)$ in the initial interval.[5] As functions $D_k e^{\alpha_k t}$ satisfy (17), the function $\sum D_k e^{\alpha_k t}$, which represents with sufficient approximation $J(t)$ in the initial interval, will be a general solu-

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[4] "Ein Schiffbauzyklus?" *Weltwirtschaftliches Archiv*, B. 34, H.1.

[5] *Loc. cit.*, p. 158.

tion of (17).[6] That solution is, of course, a real one, thus D_k and D_{-k} must be complex conjugate numbers, and $J(t)$ can be represented as follows:

$$
\begin{aligned}
J(t) = e^{(m-x_1)t/\theta}&\left(F_1 \sin y_1 \frac{t}{\theta} + G_1 \cos y_1 \frac{t}{\theta} \right) \\
(25) \\
&+ e^{(m-x_2)t/\theta}\left(F_2 \sin y_2 \frac{t}{\theta} + G_2 \cos y_2 \frac{t}{\theta} \right) \cdots .
\end{aligned}
$$

On the basis of that solution we cannot yet say anything definite about the character of fluctuations of $J(t)$, as the constants F_k and G_k depend upon the form—unknown to us—of the function $J(t)$ in the initial interval. But here we can take advantage of the following circumstance. It may be inferred from Tinbergen's argument when he solves the equation (24) that

$$
(26) \qquad x_1 < x_2, x_1 < x_3 \cdots .
$$

Let us divide $J(t)$ by:

$$
e^{(m-x_1)t/\theta}\left(F_1 \sin y_1 \frac{t}{\theta} + G_1 \cos y_1 \frac{t}{\theta} \right).
$$

According to the inequality (26), for a sufficiently great t the sum of all the expressions other than the first one will be equal to an arbitrarily small value ω:

$$
\frac{J(t)}{e^{(m-x_1)t/\theta}\left(F_1 \sin y_1 \dfrac{t}{\theta} + G_1 \cos y_1 \dfrac{t}{\theta} \right)} = 1 + \omega.
$$

At a time sufficiently distant from the initial interval, the following equation will be true with an arbitrarily small relative error:

$$
(27) \qquad J(t) = e^{(m-x_1)t/\theta}\left(F_1 \sin y_1 \frac{t}{\theta} + G_1 \cos y_1 \frac{t}{\theta} \right).
$$

That equation represents harmonic vibrations with an amplitude decreasing, constant, or increasing, according as $x_1 \gtreqless m$. Their period, and the degree of progression or degression they show, do not depend on the form of the function $J(t)$ in the initial interval. (It is worth mentioning that, as follows from Tinbergen's analysis, vibrations represented by (27) have a period longer than 2θ, while vibrations represented by the expressions on the right side of the equation [25] which we dropped, have a period shorter than θ).

[6] *Loc. cit.*, p. 157.

If now we fix the origin of the time axis so as to equate $J(t)$ from (27) to zero for $t = 0$, that equation will assume the form:

$$J(t) = F_1 e^{(m-x_1)t/\theta} \sin y_1 \frac{t}{\theta}$$

or, taking into consideration (16):

(28) $$I(t) - U = F_1 e^{(m-x_1)t/\theta} \sin y_1 \frac{t}{\theta}.$$

Case II. In that case (24) has two real roots z_1' and z_1'', among complex roots like $x_1 \pm iy$. As in the first case, we get here, for a time sufficiently distant from the initial interval:

$$J(t) = D_1' e^{(m-z_1')t/\theta} + D_1'' e^{(m-z_1'')t/\theta}.$$

It follows from that equation that there are no cyclical vibrations.

The results of the above analysis can be summarized as follows:

Cyclical variations occur in our economic system only when the following inequality is satisfied:

$$l > \frac{1}{e},$$

transformed, by putting the value of l from (20) into:

(29) $$m + \theta n > e^{m-1}.$$

As we know, m is positive (see p. 331). We can easily prove that a necessary, though insufficient, condition, at which (29) is satisfied, i.e., there are cyclical variations, is that n be positive too.

Fluctuations of I at a time sufficiently distant from the initial interval $(0, \theta)$ will be represented by the equation:

(28) $$I(t) - U = F_1 e^{(m-x_1)t/\theta} \sin y_1 \frac{t}{\theta}.$$

The amplitude of fluctuations is decreasing, remains constant, or rises, according as $x_1 \gtreqless m$.

The period is equal to

(30) $$T = \frac{2\pi}{y_1} \theta.$$

On the basis of equations

(7) $$A(t) = \frac{1}{\theta} \int_{t-\theta}^{t} I(\tau) d\tau$$

and

$$(4) \qquad\qquad L(t) = I(t - \theta)$$

we can show L and A as functions of t, and see that these values are fluctuating, like I, around the value U. K is obtained by integration of:

$$(8) \qquad\qquad K'(t) = L(t) - U.$$

It also fluctuates around a certain constant value, which we denote by K_0. The whole calculation will be given in the next chapter with respect to a particular case when the amplitude of fluctuations is constant.

III

If, while $x_1 = m$, the amplitude of fluctuations remains constant, (28) assumes the form:

$$(31) \qquad\qquad I(t) - U = a \sin y_1 \frac{t}{\theta}$$

where a is the constant amplitude.

That case is of a particular importance as it appears to be nearest to actual conditions. Indeed, in reality we do not observe any *regular* progression or degression in the intensity of cyclical fluctuations.

Putting the value of I from (31) into (7) and (4) we get

$$A - U = \frac{1}{\theta} \int_{t-\theta}^{t} \left(a \sin y_1 \frac{\tau}{\theta} + U \right) d\tau - U$$

$$(32)$$

$$= a \, \frac{\sin \dfrac{y_1}{2}}{\dfrac{y_1}{2}} \sin y_1 \frac{t - \dfrac{\theta}{2}}{\theta}$$

and

$$(33) \qquad\qquad L - U = a \sin y_1 \frac{t - \theta}{\theta}.$$

From (8) and (33)

$$K'(t) = a \sin y_1 \frac{t - \theta}{\theta}.$$

Integrating:

$$(34) \qquad K - K_0 = - a \frac{\theta}{y_1} \cos y_1 \frac{t - \theta}{\theta}$$

where K_0 is the constant around of which K is fluctuating, equal here to the average volume of the industrial equipment K during a cycle.

In a similar way, the average values of I, A, and L, during a cycle will be equal in our case of constant amplitude to the constant U around which I, A, and L, are fluctuating.

Taking into consideration the condition of a constant amplitude $z_1 = m$ we shall get now from (20) and (24):

$$(35) \qquad \cos y_1 = \frac{m}{m + \theta n}$$

and

$$(36) \qquad \frac{y_1}{tgy_1} = m.$$

These equations allow us to determine y_1; moreover, they define the interrelationship of m and n.

Between m and n there is still another dependency. They are both coefficients in the equation:

$$(11) \qquad I = m(C_1 + A) - nK$$

which must be true for one-cycle-averages of I and A equal to U, and for the average value of K equal to K_0:

$$U = m(C_1 + U) - nK_0.$$

Hence:

$$(37) \qquad n = (m - 1) \frac{U}{K_0} + m \frac{C_1}{K_0}.$$

Thus, if values of U/K_0 and C_1/K_0 were given, we could determine m and n from (35), (36), and (37). U/K_0 is nothing else but the rate of amortization, as U is equal to the demand for restoration of equipment, and K_0 is the average volume of that equipment. C_1 is the constant part of the consumption of capitalists. U/K_0 and C_1/K_0 may be roughly evaluated on the basis of statistical data. If we also knew the average gestation period of investments θ, we could determine y_1 and the duration of the cycle $T = 2\pi\theta/y$.

We evaluate the *gestation period of investments* θ on the basis of data of the German *Institut fuer Konjunkturforschung*. The lag between the curves of beginning and termination of building schemes (dwelling

houses, industrial and public buildings) can be fixed at 8 months; the lag between orders and deliveries in the machinery-making industry can be fixed at 6 months. We assume that the average duration of θ is 0.6 years.

The rate of amortization U/K_0 is evaluated on the basis of combined German and American data. On that of the German data, the ratio of amortization to the national income can be fixed at 0.08. With a certain approximation, the same is true for U.S.A. Further, according to official estimates of the wealth of U.S.A. in 1922, we set the amount of fixed capital in U.S.A. at \$120 milliards (land excepted). The national income is evaluated at \$70 milliards for 5 years about 1922. The rate of amoritization would thus be $0.08 \cdot 70/120$, i.e., *ca.* 0.05.

Most difficult is the *evaluation of* C_1/K_0. K_0 was fixed at \$120 milliards, C_1 is, as we know, the constant part of the consumption of capitalists. Let us evaluate first the average consumption of capitalists in U.S.A. in the period 1909–1918. The total net profit in that period averaged, according to King, \$16 milliards deflated to the purchasing power of 1913. The average increment of total capital in that period is estimated by King at \$5 milliards. That figure includes savings of workpeople, but, on the other hand, 16 milliards of profits cover also "capitalistic" incomes of workpeople (use of own houses, etc.). Thus, the difference, $16 - 5 = 11$ millards of 1913-dollars, represents with a sufficient degree of accuracy the consumption of capitalists (farmers included). The average national income amounted in the period 1909–1918 to \$36 milliards with the purchasing power of 1913 (King). The ratio of the consumption of capitalists to the national income would thus be 0.3. As, further, the average income during 5 years around 1922 amounted, as mentioned, to \$70 milliards of current purchasing power, the consumption of capitalists in these years may be estimated at \$21 milliards. Now, we have to determine the constant part of that consumption. In order to do that, we assume that when the volume of capitalists' gross profits deviates from the average by, say, ± 20 per cent, the corresponding relative change in their consumption is but 5 per cent, i.e., 4 times smaller. That assumption is confirmed by statistical evidence. Accordingly, the constant part of the consumption of capitalists, equal to $C_1 + \lambda B$ (see above, λ is a constant fraction, B—the total gross profit), amounts to 3/4 of \$21 milliards, i.e., to \$16 milliards. The ratio C_1/K_0 would then be 16/120 or *ca.* 0.13.

Equations (35), (36) and (37), if we put:

$$\theta = 0.6; \qquad \frac{U}{K_0} = 0.05; \qquad \frac{C_1}{K_0} = 0.13;$$

give:

$$\cos y_1 = \frac{m}{m + 0.6n}$$

$$\frac{y_1}{tg y_1} = m$$

$$n = 0.05(m - 1) + 0.13m.$$

The solution of these equations gives:

$$m = 0.95; \qquad n = 0.121; \qquad y_1 = 0.378.$$

Thus, the duration of the cycle is:

$$T = \frac{2\pi}{y_1} \theta = \frac{2\pi}{0.378} \cdot 0.6 = 10.0.$$

The figure of 10 years thus obtained as the time of duration of a cycle is supported by statistical evidence: 8 to 12 years.[7] It may be objected that values θ_1 U/K_0, C_1/K_0, on which our calculation was based, were but roughly estimated, and that the conformity between facts and theory can be merely a coincidence. Let us calculate T for such values of θ_1 U/K_0, C_1/K_0 as would be quite different from those previously taken:

θ	$\dfrac{U_0}{K_0}$	$\dfrac{C_1}{K_0}$	T
0.6	0.05	0.13	10.0
0.6	0.03	0.13	10.0
0.6	0.07	0.13	10.0
0.6	0.05	0.07	13.2
0.6	0.05	0.19	8.5
0.3	0.05	0.13	7.1
0.9	0.05	0.13	12.5

We see that the value of U/K_0 plays no great rôle with respect to the result of our calculation. We see further that when values of C_1/K_0 and θ differ by almost 50 per cent from those adopted before (C_1/K_0 =0.13 and U/K_0=0.05) solutions for T move between 7 and 13 years. The actual duration of the cycle being, as already mentioned, 8 to 12

[7] Shorter cycles can be considered as "short-wave" fluctuations.

years, we can safely say that, irrespective of the degree of accuracy in estimating θ, U/K_0, C_1/K_0, there is no flagrant incongruity between the consequences of our theory and reality.

There is one more question to be dealt with. During the whole time we considered, as stated at the very beginning of the study, an economic system free of secular trend. But a case when the trend is uniform, and when gross accumulation, consumption of capitalists, and the volume of industrial equipment, show the same rate of development, can be easily reduced to a state "free of trend" simply by dividing all these values by the denominator of the trend. Interrelationships stated in our chapter I will remain true for these quotients, with the following changes: (1) The value U will be no longer equal to the demand for restoration of the used-up equipment, but it will cover as well the steady demand for the expansion of the existing equipment as a result of the uniform secular trend. Thus U/K_0 will be equal not to the rate of amortization 0.05, but, assuming the rate of net accumulation equal, say, to 3 per cent, to 0.08. (2) Also stocks of goods, previously considered constant, will increase in the same proportion under the influence of the trend. That steady increment of stocks per unit of time—let us call it C_2—will be a component of the gross profit B, now equal to $C+C_2+A$, where C is the personal consumption of capitalists, C_1 the steady increment of stocks, and A the production of capital goods. If we now consider that, according to equation (2), the consumption of capitalists C is equal to $C_1+\lambda B$, we see that B is proportionate to C_1+C_2+A. The constant C_1+C_2 will play in our considerations the same rôle as C_1 previously did. According to the official estimate of the national wealth of the U.S.A., the volume of stocks of goods amounts to 0.3 of the volume of the industrial equipment, i.e., to $0.3 \cdot K_0$. If the rate of net accumulation be 3 per cent, C_2 will be $0.03 \cdot 0.3 \cdot K_0$. Hence, instead of $C_1/K_0=0.13$ we must take $(C_1+C_2)/K_0=0.14$. From the above table we may easily see that both modifications—0.08 instead of 0.05 for U/K_0 and 0.14 instead of 0.13 for C_1/K_0,—will have but little effect on the result of the calculation of T.

We shall now determine, on the basis of (31), (32), (33), and (34), equations of curves I, A, L, and K, with $\theta=0.6$ and $T=10.0$:

$I-U=a \sin 0.63t$
$A-U=0.98a \sin 0.63 \, (t-0.3)$
$L-U=a \sin 0.63 \, (t-0.6)$
$K-K_0=-1.59a \cos 0.63 \, (t-0.6).$

Assuming, in conformity with the above estimate, $U/K_0=0.05$, we find the following formulae for the relative deviations from the state of equilibrium:

$$(38) \qquad \frac{I - U}{U} = \frac{a}{U} \sin 0.63t$$

$$(39) \qquad \frac{A - U}{U} = \frac{a}{U} \cdot 0.98 \cdot \sin 0.63(t - 0.3)$$

$$(40) \qquad \frac{L - U}{U} = \frac{a}{U} \sin 0.63(t - 0.6)$$

$$(41) \qquad \frac{K - K_0}{K_0} = -\frac{a}{U} \cdot 0.08 \cos 0.63(t - 0.6).$$

IV

Figure 2 represents the curves of investment orders I, of production of capital goods A, of deliveries of industrial equipment L, and of the volume of industrial equipment K, which correspond to the formulae (38), (39), (40), and (41).
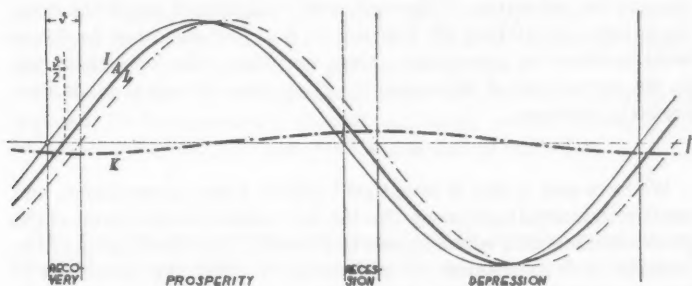
FIGURE 2

Let us recall the dependence (11), $I = m(C_1 + A) - nK$, wherefrom it follows, if m and n are positive (see p. 333), that the volume of investment orders is an increasing function of the gross accumulation equal to the production of capital goods, and a decreasing function of the volume of the existing industrial equipment. Having these in mind, we can, on the basis of the Figure 2, explain the mechanism of the business cycle.

Recovery is the phase of the cycle of a duration θ, when the volume of investment orders begins to exceed the volume of the demand for restoration of the industrial equipment. But the very volume of the existing industrial equipment is not yet increasing, as deliveries of new equipment still remain below the demand for restoration of equipment.

The output of capital goods A, equal to the gross accumulation, is on the increase. Meanwhile, the volume of the existing industrial equipment K is still on the decrease, and, as a result, investment orders rise at a rapid pace.

During *prosperity* also deliveries of equipment exceed the demand for restoration of the equipment, thus the volume of the existing equipment is increasing. The rise of K at first hampers the rise of investment orders, and at last causes their drop. The output of capital goods follows suit, and begins to fall off in the second part of prosperity.

During *recession* investment orders are below the level of the demand for restoration of the industrial equipment, but the volume of the existing industrial equipment K is still on the increase, since deliveries are still below the demand for restoration. As the volume of production of capital goods, equal to the gross accumulation A, continues to fall off, the volume of investment orders I is decreasing rapidly.

During *depression* deliveries of equipment are below the level of the demand for restoration of the equipment, and the volume of the existing equipment is falling off. The drop in K at first smoothes the downward tendency in investment orders, and then calls forth their rise. In the second part of depression the production of capital goods, too, begins to increase.

*   *   *

We have seen a plot of investment orders, gross accumulation, and existing industrial equipment. But the fluctuations in the volume of the gross accumulation, which appear as a result of the functioning of the business cycle mechanism, must necessarily affect the movement of prices and the total volume of production. Indeed, the real gross profit B is, on the one hand, an increasing function of the gross accumulation A (B being proportionate to C_1+A, where C_1 is the constant part of the consumption of capitalists, see above) and, on the other hand, it can be represented as a product of the general volume of production and of the profit per unit of production. In that way, the general volume of production and prices (or rather the ratio of prices to wages determining the profit per unit) rises in the upward part of the cycle as the gross accumulation increases.

The interdependence of gross accumulation, equal to the production of capital goods, and of the general movement of production and prices, is realized in the following way. While the output of capital goods increases by a certain amount, in the general volume of production, beside that increment, there is another increment because of the increased demand for consumers' goods on the part of workers recently

hired by industries making capital goods.[8] The consequent increase in employment in industries making consumers' goods results, in its turn, in an increase in the demand for consumers' goods on the part of workers. As simultaneously there is an advance of prices, the new demand is but partly met by the new production. The remaining part of that demand is satisfied at the expense of the "old" workers, whose real earnings suffer a reduction. The general level of production and prices must eventually rise, so as to provide for an increment of the real profit equal to the increment of the production of capital goods.

That description is incomplete in so far as it does not reckon with changes in the personal consumption of capitalists. That consumption—C—is dependent, to a certain extent, on the proportions of the total profit B, and increases in accordance with the gross accumulation A (from equations (2) and (3) it follows that $C = (C_1 + \lambda A)/(1 + \lambda)$, where λ is a constant fraction). The increase in the consumption of capitalists has the same effect as the increase in production of capital goods: there is an increase in the volume of production of consumers' goods for the use of capitalists; as a result, employment increases, hence an additional demand for consumers' goods for the use of workers, and, eventually, a further rise of production and prices.

The general level of production and prices must rise, eventually, so as to provide for an increment of the real profit equal to the increment of the production of capital goods and of the consumption of capitalists.

*   *   *

The question may arise wherefrom capitalists take the means to increase at the same time the production of capital goods and their own consumption. Disregarding the technical side of the money market such as, e.g., the variable demand for means of payment, we may say that these outlays are "financing themselves." Imagine, for instance, that some capitalists withdraw during a year a certain amount from their savings deposits, or borrow that amount at the Central Bank, in order to invest it in the construction of some additional equipment. In the course of the same year that amount will be received by other capitalists under the form of profits (since, according to our assumptions, workers do not save), and put again into a bank as a savings deposit or used to pay off a debt to the Central Bank. Thus, the circle will close itself.

Yet in reality, just because of the technical side of the money market, which, as a matter of fact, forms its very nature, a credit inflation becomes necessary for two reasons.

[8] We take for granted that there is a reserve army of unemployed.

The first is the fact of the curve I of investment orders not coinciding exactly with that of production of capital goods A, equal to the gross accumulation. When giving an investment order, the entrepreneur has to provide first some corresponding fund, out of which he will currently finance the filling of that order. At any time the corresponding bank account will be increased (per unit of time) by the amount I equal to the volume of orders allocated, and simultaneously decrease by an amount A spent on the production of capital goods.[9]

In that way, at any time the investment activities require an amount I (per unit of time), notably: $I-A$ to form new investment reserves, and A to be spent on the production of capital goods. The actually spent amount A "finances itself," i.e., comes back to the bank under the form of realized profits, while the increment of investment reserves $I-A$ is to be created by means of a credit inflation.

Another reason for the inflation of credit is the circumstance that the increase in the production of capital goods or in the consumption of capitalists, i.e., increased profits, calls forth a rise of the general level of production and prices. This has the effect of increasing the demand for means of payment under the form of cash or current accounts, and to meet that increased demand a credit inflation becomes necessary.

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[9] The values concerned are not exactly the real values of I and A but corresponding amounts of money, calculated at current prices.

MEETINGS OF THE ECONOMETRIC SOCIETY
IN CHICAGO AND PITTSBURG
DECEMBER, 1934

AMERICAN meetings of the Econometric Society were held on December 27-28, 1934, in Chicago in connection with the annual meetings of the Social Science Societies, and in Pittsburgh on December 28-29 jointly with Section K of the American Association for the Advancement of Science.

On Thursday morning, December 27, at 9:30 A.M., the opening session of the Chicago meeting was held in the Palmer House, the general topic being "Elasticity of Demand and Competition." Theodore O. Yntema of the University of Chicago presided. Hans Neissen of the University of Pennsylvania began the meeting with a paper, "The Average Elasticity of Demand." Dr. Neissen denied that the average elasticity of a single income receiver's demand schedule (as measured in the actual prices paid at a certain income) is equal to unity, declaring that no conclusion about this average elasticity could be derived from the evident correlation between variations in the outlay for different commodities.

The second paper of this session, "On the Elasticity of Substitution between Labor and Capital," was presented by Fritz Machlup of Vienna, who said, in part: "The elasticity of substitution is a useful tool in the analysis of 'derived demand', i.e., in the analysis of the demand for factors of production. The concepts of elasticity of substitution defined independently by J. R. Hicks and Joan Robinson were mistaken for one another. Robinson's concept can not lead to deduction of anything regarding the final demand for factors, while Hicks' bears a definitive functional connection with the elasticity of demand for factors. In other words, the latter is based on technical conditions of the substitution of one factor for another within a single industry, the former on the rearrangement of the factors of the whole system following an increase in the quantity of factors."

The third paper was given by Oskar Lange of the University of Krakow, Poland, whose subject was "A Generalized Theory of Limited Competition." After this paper, there was discussion led by E. H. Chamberlin of Harvard University, Mordecai Ezekiel of the Department of Agriculture, and Theodore O. Yntema of the University of Chicago.

At 11:00 A.M. the general topic, "Demand and Supply Functions," was considered at a meeting presided over by Charles F. Roos of Colorado College. L. H. Bean of the U. S. Department of Agriculture spoke

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first on "The Use of Demand and Supply Curves in Economic Planning." He showed in an interesting fashion the various uses to which demand and supply curves had been put in inaugurating and carrying through the Administration's program of agricultural adjustment. In addition, he presented a demand law for new automobiles. His paper was followed by one on "Inter-textile Competition" by Victor von Szeliski of the National Recovery Administration, after which T. O. Yntema and Joseph A. Schumpeter led a discussion.

The Econometric Society held a joint session with the American Statistical Association on the afternoon of the same day at the Palmer House. Irving Fisher of Yale University presided and the general topic considered was "Monetary Policy and Price Changes During Recovery: A Survey of Relevant Evidence." The opening paper, entitled "Recent Monetary Experiments and Their Effect upon the Theory of Money and Price," was given by Willford I. King of New York University. He concluded: "(1) Changing the price of gold affects immediately and almost proportionally the paper prices of commodities entering largely into international trade. However, it does not affect the gold prices of such commodities. It has little immediate effect upon the prices of goods sold mainly in the domestic market. (2) Depreciation of the paper currency of a nation has little or no effect upon either the physical volume of its foreign trade or upon its balance of trade. (3) Increases in the price level may actually occur through increases in velocity of circulation even though the volume of circulating medium remains unchanged. (4) The price level can be definitely controlled by means of 'open market' operations. However, this point requires further experimental evidence before it can be considered as established inductively."

Frank D. Graham of Princeton University was the next speaker. Treating the subject "Recent Movements in International Price Levels and the Doctrine of Purchasing Power Parity," he said, in part, "The monetary policy pursued by many important countries in the last few years necessitates the adjustment of prices to exchange rates if the world integration of prices assumed in the doctrine of purchasing power parity is to be preserved. This reverses the ordinary sequence of events. Since prices are much less mobile than free exchange rates, and the great bloc of countries with currencies devalued from 40 to 50 per cent dominate international markets, the adjustment has not as yet gone far and, to the degree that it has occurred, has been thrust beyond the borders of those countries. The result is that there are three distinct price levels in the world at present, that of the gold bloc, that of the countries with 40–50 per cent depreciation, and that of the countries which have carried devaluation considerably beyond this

point. The resistance to deflation in the countries of the gold bloc and the halting rise in prices in the countries of devaluation seem likely to lead to a perpetuation of this situation through a regimentation of trade in the best traditions of Mercantilism. The question may well be raised whether, in the interest of international trade, we should not divert attention from the mere stabilization of exchange rates to the attainment of an exchange structure which will provide parity of purchasing power to important currencies on the basis of such price levels as the several countries may desire."

The final paper of the session, "Price Inflexibility and the Requirements of a Stabilizing Monetary Policy," was presented by Gardiner C. Means of the Department of Agriculture. This has since been printed as a U. S. Senate Document. Following Means' paper, there was a lively discussion, participated in by George F. Warren of Cornell University, Harry Gideonse of the University of Chicago, and Charles F. Roos.

On Thursday evening, December 27, at 8:00 P.M., in the Sky Room of the Palmer House, a session was devoted to "Securities and Investments" with Irving Fisher presiding. Victor von Szeliski discussed "Some Stock Market Price-Volume Relationships," treating certain characteristic patterns of action in stock prices that appear when figures on price and volume are adjusted to allow for changes of different magnitude.

Henry C. Murphy of Investment Counsel, Inc. of Detroit, Michigan, and Henry T. Bodman of the National Bank of Detroit next presented a paper on "Statistical Factors Affecting Public Utility Bond Yields." The statistical factors considered were (1) charge coverage, (2) the ratio of net income to gross revenues, (3) the ratio of gross revenues to funded debt, and (4) the ratio of stock equity to funded debt. These factors included, directly or indirectly, all the income statement and balance sheet ratios generally used in public utility bond analysis. The simple coefficients of correlation between each of these factors and yield for the sample studied were found to be as follows: (1) $-.893$, (2) $-.588$, (3) $-.763$, and (4) $-.283$. The ratio of stock equity to funded debt thus seemed of negligible influence. When charge coverage was held constant, factor (2) could be favorable only in the exact proportion that factor (3) was unfavorable, and *vice versa*. This was demonstrated algebraically and confirmed by the computation of partial coefficients. Since the market laid greater stress on a small debt (factor 3) than on a large proportion of gross saved for net (factor 2), the important conclusion followed that, as between bonds with the same charge coverage, a high net income was an actual handicap. Murphy and Bodman expressed the opinion that the ratio

of net income to gross revenues was the factor which properly ought to be stressed most, but reserved discussion of this for a later paper.

On Friday, December 28, at 10 A.M., the Econometric Society held a joint meeting with the American Economic Association and the American Statistical Association at the Palmer House to celebrate the Walras Centennial. Joseph A. Schumpeter of Harvard University presided. In opening the program, he said, "Like every other science, economics presents innumerable problems and aspects, each appealing to different aptitudes and different types of mind. But, whatever the aspect or the approach, the name of Léon Walras towers above all others. Both in vision and achievement he was, without doubt, the greatest of economic theorists. His work is more alive today than ever before. It is exercising a greater influence and it is serving as a starting point for growing minds more than it ever did during the life-time of the author. Though we have today travelled far from the fountain head, it is surprising how often all of us turn back to Walras' writings. Let us for a moment recall with gratitude the Committee of Seven, and particularly M. Ruchonnet and Professor Dameth, on whose recommendation the chair of political economy at Lausanne was given to Walras in 1870. Whoever knows—and who can help knowing—with what doubt and resistance exact theory is received both inside and outside of the academic world, can not fail to admire the open-mindedness and the courage of the four members of the Committee who approved of Walras. They rendered a service of historic importance to our science."

William Jaffé of Northwestern University then spoke on "Some Letters and Unpublished Papers of Léon Walras," following which Arthur W. Marget presented a paper, "The Monetary Aspects of the Walrasian System." Both these addresses appear in full in *The Journal of Political Economy*, Vol. XLIII, Number 2, April 1935.

Dr. Oskar Lange of the University of Krakow, Poland, opened up the discussion saying, in sum: "Léon Walras is best known as one of the inventors of the marginal utility concept, along with Menger and Jevons. But this hardly does justice to the greatness of his achievement, which lies in his development of the concept and a theory of *general* economic equilibrium. He introduced the marginal utility concept without abandoning the major contribution of the labor-value theory. He proved himself to be a great synthesist by combining the contribution of classical economics with the idea that demand is determined by marginal utility. This synthesis is found in his mathematical theory of general economic equilibrium."

Finally, Professor Irving Fisher closed the discussion with the remarks: "Instead of commenting specifically on the two excellent papers of Professor Jaffé and Professor Marget, I will speak of my personal

history in relation to Walras' work and mathematical economics generally. In college (1884–1888) I was interested primarily in mathematics. In the Graduate School (1888–1891) I continued those studies under J. Willard Gibbs, but, because of Professor William Graham Sumner's outstanding personality, I also studied Sumnerian economics. This was before Sumner turned to 'societology.' After a year or more of studies split about evenly between economics and mathematics, I said to Professor Sumner one day, 'I am getting worried about selecting my thesis for a Ph.D. I don't know whether to choose mathematics or economics.' Professor Sumner said, 'Why not mathematical economics?' I said, 'I have never heard of such a field.' He referred me to the literature—Jevons, Walras, and a book just out, *Untersuchungen über die Theorie des Preises*, by Auspitz and Lieben of Vienna. I read the last-named book with the utmost thoroughness and it has always seemed to me to be an important contribution. The result of these various studies was my doctor's thesis, *Mathematical Investigations in the Theory of Value and Prices*. In 1893–1894 I spent a year in Europe on my wedding trip. I saw many famous economists, including Edgeworth, Menger, Lieben, Pantaleoni, Pareto, and Walras. Walras was not well. Pointing to his bald head he said, 'My head won't work.' His wife spoke of me to Mrs. Fisher as one of her husband's 'disciples.' He told me we must 'stand together.' But he spoke disparagingly of Auspitz and Lieben, which gave me a slightly unfavorable impression of him. Later, in Oxford, I asked Edgeworth if Marshall had reference to Walras when he spoke of 'lengthy translations of economics into mathematics,' and he answered, 'Yes.' This led me further to discount Walras, so that I neglected to read him beyond what had been germane to value theory. I had only a vague idea of his monetary theories. I did translate, or guided the translation of, one of Walras' monographs—on the geometrical determination of prices, published in the *Annals of the Academy of Political and Social Sciences*. But it is clear to me now that I have never done full justice to Walras as compared with Cournot, whose main work I helped translate and annotate. But, even so, I have always recognized the fact that Walras was one of the greatest mathematical economists. After I had my first enthusiasm over Walras and mathematical economics, I soon found that there was only a very limited market for such wares and almost despaired of ever seeing economics grow into a true science such as we now call econometrics. But now I am thrilled with a new enthusiasm. Even 'marginal utility' or 'rareté,' the idea of which originated with Walras (among others), has now become the subject of measurement. I tried to postulate the conditions for this in my doctor's thesis and later returned to the subject in a short monograph. Ragnar Frisch of the University of Norway

had independently developed a different method and later synthesized the two in a very able book. This is merely one example of how the truth planted by Walras is today growing into a rich harvest of econometrics. As I relinquish the Presidency of the Econometric Society, it is a great satisfaction to feel that the work done by Walras and others in mathematical economics is now bearing more fruit than ever."

With the Walras Centennial Program, the Chicago meeting came to an end.

* * * * *

The meetings of the Econometric Society at Pittsburgh were held on December 28–29, some of the sessions being held jointly with Sections A, K, and M, of the American Association for the Advancement of Science.

Salient features of the papers at those meetings were reported in *Science*, Vol. LXXXI, No. 2092 (February 1, 1935). The papers "Economic Theory of the Shorter Work Week," by Charles F. Roos, and "Competition under Secret and Open Prices," by Simon N. Whitney, were published in full in ECONOMETRICA, III, No. 1 (January, 1935). Some further notice of a few papers seems desirable here.

In a paper, "Mathematical Relations Between Rental Quotations and Other Economic Factors," by H. Pixley of Wayne University, the problem of a statistical analysis of the measurable factors involved in rental levels of cities was attempted. The principal source of data used was the U. S. Census reports on median rentals in 1930. This yielded a rental figure for all cities over 2500, the only source to give such complete coverage. The regions covered were Southern Michigan and Northern Ohio. The principal measurable factors were found to be, in the order of importance, size of city, and rate of increase in the preceding decade. By fitting a formula of the type $y = a + b \log x + cw$, where y is rental in dollars per month, x is population in thousands, and w is the increase of the preceding decade expressed as a percentage of the population at the beginning of the decade, theoretical rental values were obtained which varied from the actual values by less than $4.00 in over three fourths of the cases, while the mean variation was about $2.75. He concluded that in most cases the unmeasurable factors should not account for more than 20 per cent of the local rental level and in most of these cases it was less than 10 per cent. Rent was not broken down statistically into its components, but Pixley felt that the two most important were measured, at least approximately, in the two regions concerned.

H. J. Titus of the Franklin Railway Supply Company of New York spoke next on the subject, "Theoretical and Statistical Studies of Loco-

motive Maintenance Expense and Replacement." In reviewing the major conditions which affected maintenance expense of locomotives, he brought out that, by confining a study to an individual road, the number of causes for variability might be greatly reduced. The major causes affecting such expenses on a given road would be miles between general repairs, miles per month, power of the locomotive, and age. By the use of Criterion I as developed by W. A. Shewhart, tests were made to determine whether or not the data were predictable. These tests were shown by tables and charts for one class of locomotive. Equations for maintenance expense, as well as for miles between general repairs and miles per month, were obtained in terms of the variables most affecting them. Charts were presented showing the variations of expenses with miles between general repairs, power, and age. The results indicated the value of using the locomotive with the minimum power from the main cylinders for a given operation, obtaining the total by using supplemental power. Also, it was possible to determine the most effective means of reducing maintenance on existing locomotives, as well as determining when these locomotives became uneconomical for a specific operation.

Walter Keim, of the National Recovery Administration, read the concluding paper, entitled "The Price Pattern of Recovery," saying, in part: "In a period of declining prices, the predominating forces at work causing a general price recession are not equally effective on each commodity or group of prices. But, when the forces operating to raise prices have achieved a predominating rôle, the success of the resulting price movement appears to be, on the average, in proportion to the extent of the price decline. The decline of 37.2 per cent from June, 1929 to February, 1933, in the index number representing the general wholesale price level in the United States, was the result of a 49.9 per cent decline in raw material prices, a 39.1 per cent decrease in the prices of semi-finished goods, and a 30.8 per cent reduction in finished goods prices. In the first recovery period, February to August, 1933, semi-manufactured goods regained more than 40 per cent of the price drop of the preceding four years, as compared with raw materials and finished goods, which exhibited more conservative rates of recovery—approximately 26 per cent each. At the end of the second phase of the recovery movement, March, 1934, semi-finished products had recovered to a point very little above the previous recording; however, prices of raw materials and finished goods, during this period, had advanced to "recovery" levels of 36.3 per cent and 39.3 per cent, respectively. The most recent phase of the recovery has been characterized by normal increases in prices of raw materials and finished goods, principally caused by upward movements in farm product and food prices; the

average price for semi-finished products registered a reduction during
this period to the extent that the index number for this class of goods
is lower than it was in August, 1933. To date, the index number for
"All Commodities" has reached a point indicative that prices, on the
average, have recovered nearly 50 per cent of what they lost during
the "decline" period. A description of the frequency of price move-
ment for each of the three classes of goods lends supporting evidence to
the foregoing analysis made by other methods concerning the insta-
bility of raw material prices. Semi-manufactured goods, however,
appeared decidedly more stable than raw materials, so far as the re-
lative number of price changes was concerned. Since 1929, the fre-
quency of price changes for all classes of goods tended to show a
marked increase as the depression price decline continued to lower
and lower levels. The forecasts, however, point toward an increased
stability of prices in 1935 for each of these groups, and this may be
regarded as an indication that the frequency of price movement is
approaching a comparative normal."

                                                    CHARLES F. ROOS

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ERRATUM

In the April 1935 issue of ECONOMETRICA, "Stable Prices vs. Stable
Exchanges", by C. Reinold Noyes, page 131, the twelfth line from the
bottom of the page should read:
  ". . . would be to increase the quantity, . . ."

THE MATHEMATICAL DISTRIBUTIONS USED IN
IN THE COMMON TESTS OF SIGNIFICANCE

By R. A. Fisher

Introduction.—The three frequency distributions which provide the greatest number of tests of significance in common use are all closely related. The main types of application will be found illustrated arithmetically in the author's book *Statistical Methods for Research Workers* and in other publications in which extensive use is made of the arithmetical arrangement known as the Analysis of Variance. Some need has, however, been felt by mathematicians for a concise account of the algebraic properties and relationships of these distributions, and the following are essentially lecture notes designed to give a mathematical student a clear account of their properties.

1. The frequency distribution of χ^2.—If $x_1, x_2, \cdots, x_n$, are independent values of a variate distributed normally about zero, with unit variance, then the quantity

$$\chi^2 = S(x^2),$$

where S, as usual, stands for summation over the sample, has a distribution given by:—

$$df = \frac{1}{\frac{n-2}{2}} (\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-2)} e^{-\frac{1}{2}\chi^2} d(\tfrac{1}{2}\chi^2).$$

This may be proved in several ways, two of which deserve notice. (a) By induction, for $n = 1$, the expression reduces to

$$\sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}x^2} dx,$$

which is clearly the distribution of x^2 for a single observation. If, now, $2u$ is the sum of the squares of n independent values of the variate, and has the distribution,

$$df = \frac{1}{\frac{n-2}{2}!} u^{\frac{1}{2}(n-2)} e^{-u} du,$$

and x is an additional observation independent of the others, then

$$\chi^2 = 2u + x^2,$$

and its distribution is to be inferred from the simultaneous distribution

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$$df = \frac{1}{\frac{n-2}{2}!} \cdot \sqrt{\frac{2}{\pi}} \, u^{\frac{1}{2}(n-2)} e^{-u-\frac{1}{2}x^2} du\, dx.$$

If we now substitute

$$u = \tfrac{1}{2}(\chi^2 - x^2), \qquad du = d(\tfrac{1}{2}\chi^2),$$

we have

$$df = \frac{1}{\frac{n-2}{2}!} \sqrt{\frac{2}{\pi}} \, e^{-\frac{1}{2}x^2} d(\tfrac{1}{2}\chi^2) \cdot \left(\frac{\chi^2 - x^2}{2}\right)^{\frac{1}{2}(n-2)} dx,$$

in which x takes all values from 0 to χ. Integration with respect to x will, therefore, yield a factor χ^{n-1} or $(\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-1)}$ (with a constant which need not be determined, but which may be obtained from the Eulerian integral of the first kind), giving the distribution

$$df = \frac{1}{\frac{n-1}{2}!} (\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-1)} e^{-\frac{1}{2}\chi^2} d(\tfrac{1}{2}\chi^2),$$

in accordance with the general formula.

Although the proof by induction is an attractive exercise in Eulerian integrals, many students find an alternative proof based on Euclidean hyperspace more simple and direct.

If $x_1 \cdots x_n$ are the co-ordinates of a point in such space, the frequency density at any point is proportional to $e^{-\frac{1}{2}x^2}$, and depends only on the distance of the sample point from the origin. The region in which this density exceeds any specified value is, therefore, a hypersphere in n dimensions having volume proportional to χ^n. The volume in which χ lies within any elementary range $d\chi$ is, therefore, proportional to

$$\chi^{n-1} d\chi,$$

and the element of frequency in this range is proportional to

$$\chi^{n-1} e^{-\frac{1}{2}\chi^2} d\chi.$$

The Eulerian integral of the second kind,

$$\int_0^\infty t^p e^{-t} dt = p!,$$

then supplies the required constant factor and establishes the distribution of χ or χ^2.[1]

2. *The distribution of Student's t.*—If we have a value of χ^2 derived from n independent values, and an additional value x independent of the others, "Student's t" may be defined as

$$t = \frac{x\sqrt{n}}{\chi},$$

for n degrees of freedom. Writing down the simultaneous distribution of χ and x, as above, and substituting for x in terms of t, we obtain

$$df = \frac{1}{\dfrac{n-2}{2}!}\sqrt{\frac{2}{\pi}}\cdot(\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-2)}e^{-\frac{1}{2}\chi^2(1+t^2/n)}d(\tfrac{1}{2}\chi^2)\frac{\chi}{\sqrt{n}}\,dt;$$

or, putting u for $\tfrac{1}{2}\chi^2\left(1+\dfrac{t^2}{n}\right)$,

$$\frac{1}{\dfrac{n-2}{2}!}\frac{2}{\sqrt{n\pi}}\left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}u^{\frac{1}{2}(n-1)}e^{-u}du\,dt.$$

Integration with respect to u from 0 to infinity, recollecting that t must be equally frequently positive or negative, yields the distribution found by "Student" in 1908:—

$$\frac{\dfrac{n-1}{2}!}{\dfrac{n-2}{2}!\sqrt{\pi}}\frac{dt}{\left(1+\dfrac{t^2}{n}\right)^{\frac{1}{2}(n+1)}}.$$

3. *The distribution of z.*—In the most general case arising in the analysis of variance, we consider two quantities, χ_1^2, and χ_2^2, based respectively on n_1 and n_2 values of the variate, all of which are independent. We may then define z so that

$$e^{2z} = u = \frac{n_2\chi_1^2}{n_1\chi_2^2},$$

and proceed to find the distribution of z. This is carried out, as in the previous cases, by writing down a simultaneous distribution of χ_1 and χ_2 and making the substitution

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[1] This distribution was first given by Helmert in 1875; it was later found independently by Pearson, "Student," and others, in the examination of various special problems belonging to the wide class in which it occurs.

$$\chi_1^2 = \frac{n_1}{n_2} \chi_2^2 u.$$

The integration proceeds as before, yielding the general distribution for z,

$$df = 2\,\frac{\dfrac{n_1 + n_2 - 2}{2}!}{\dfrac{n_1 - 2}{2}!\,\dfrac{n_2 - 2}{2}!}\,n_1^{\frac{1}{2}n_1}n_2^{\frac{1}{2}n_2}\frac{e^{n_1 z}dz}{(n_1 e^{2z} + n_2)^{\frac{1}{2}(n_1 + n_2)}},$$

which is evidently that of the natural logarithm of the ratio between two independent estimates of the same standard deviation based on n_1 and n_2 degrees of freedom, respectively. The wide class of problem for which z provides the appropriate test of significance is most easily recognized from this property.

4. *The Probability integral of* χ^2. The probability integral of the χ^2 distribution,

$$df = \frac{1}{\dfrac{n - 2}{2}!}\,(\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-2)}e^{-\frac{1}{2}\chi^2}d(\tfrac{1}{2}\chi^2),$$

is

$$P = \int_{\frac{1}{2}\chi^2}^{\infty}\frac{1}{\dfrac{n - 2}{2}!}\,t^{\frac{1}{2}(n-2)}e^{-t}dt,$$

which represents the probability of exceeding a given value of χ^2. Now, integrating by parts,

$$\int_{\frac{1}{2}\chi^2}^{\infty}\frac{1}{r!}\,t^r e^{-t}dt = \left[-\frac{1}{r!}\,t^r e^{-t}\right]_{\frac{1}{2}\chi^2}^{\infty} + \int_{\frac{1}{2}\chi^2}^{\infty}\frac{1}{(r-1)!}\,t^{r-1}e^{-t}dt$$

$$= \frac{1}{r!}\,(\tfrac{1}{2}\chi^2)^r e^{-\frac{1}{2}\chi^2} + \int_{\frac{1}{2}\chi^2}^{\infty}\frac{1}{(r-1)!}\,t^{r-1}e^{-t}dt.$$

When n is even, this process terminates, yielding the formula

$$P = e^{-\frac{1}{2}\chi^2}\left\{1 + \tfrac{1}{2}\chi^2 + \frac{1}{2!}(\tfrac{1}{2}\chi^2)^2 + \cdots + \frac{1}{\dfrac{n-2}{2}!}(\tfrac{1}{2}\chi^2)^{\,n-2/2}\right\}$$

Thus, for

$$n = 2, \qquad P = e^{-\frac{1}{2}\chi^2},$$
$$n = 4, \qquad P = e^{-\frac{1}{2}\chi^2}(1 + \tfrac{1}{2}\chi^2),$$

$$n = 6, \qquad P = e^{-\frac{1}{2}\chi^2}\left\{1 + \tfrac{1}{2}\chi^2 + \frac{1}{2!}\left(\tfrac{1}{2}\chi^2\right)^2\right\},$$

$$n = 8, \qquad P = e^{-\frac{1}{2}\chi^2}\left\{1 + \tfrac{1}{2}\chi^2 + \frac{1}{2!}\left(\tfrac{1}{2}\chi^2\right) + \frac{1}{3!}\left(\tfrac{1}{2}\chi^2\right)^3\right\},$$

all of which are easily calculated for a given value of $\tfrac{1}{2}\chi^2$.

When n is odd, the same process may be applied, terminating at $r = \tfrac{1}{2}$; we then have the formula,

$$P = \int_{\frac{1}{2}\chi^2}^{\infty} \frac{1}{(-\frac{1}{2})!}\, t^{-\frac{1}{2}}e^{-t}dt + e^{-\frac{1}{2}\chi^2}\left\{\frac{1}{\frac{1}{2}!}\left(\tfrac{1}{2}\chi^2\right)^{\frac{1}{2}} + \frac{1}{\dfrac{3}{2}!}\left(\tfrac{1}{2}\chi^2\right)^{\frac{3}{2}}\right.$$

$$\left. + \cdots \frac{1}{\dfrac{n-2}{2}!}\left(\tfrac{1}{2}\chi^2\right)^{\frac{1}{2}(n-2)}\right\}.$$

In the integral, write $\tfrac{1}{2}x^2$ for t, and substitute for the fractional factorials using $(-\tfrac{1}{2})! = \sqrt{\pi}$, and we find

$$P = \sqrt{\frac{2}{\pi}}\int_x^{\infty} e^{-\frac{1}{2}x^2}dx + \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}\chi^2}\left\{x + \tfrac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5\right.$$

$$\left. + \cdots \frac{1}{3\cdot 5\cdots(n-2)}x^{n-2}\right\}.$$

The integral is the familiar probability integral of the normal curve, the contribution to P being the total frequency outside the limits $\pm\chi$ times the standard deviation. The series is easily evaluated as before.

5. *Relation of the χ^2 distribution to the Poisson series.*—It will be noticed that, when n is even, the probability of the variate $\tfrac{1}{2}\chi^2$ exceeding any specified value m is

$$e^{-m}\left(1 + m + \frac{m^2}{2!} + \cdots + \frac{m^{\frac{1}{2}(n-2)}}{\dfrac{n-2}{2}!}\right),$$

which is the sum of the first $\tfrac{1}{2}n$ terms of the Poisson series, having the parameter m, or, in other words, the probability that a variate distributed in such a series is less than $\tfrac{1}{2}n$. This identity is expressed in the formula,

$$\int_m^{\infty} \frac{1}{p!}t^p e^{-t}dt = \sum_{x=0}^{p} \frac{1}{x!}m^x e^{-m},$$

where p, which takes the place of $\frac{1}{2}(n-2)$, is a positive integer or zero.

Thus, a table of χ^2 can be used as a table of the partial sum of the Poisson series; in particular, the 5 per cent value of χ^2, which is the value exceeded once in 20 trials, gives (on halving) the value of m, the "expectation" of the Poisson series of which the first $\frac{1}{2}n$ terms occupy 5 per cent of the frequency.

For example, if n is 8, the 5 per cent value of χ^2 is 15.507; consequently, we may infer that, if a rare event has been observed only $3[=\frac{1}{2}(n-2)]$ times, the observation has departed significantly from any expectation exceeding 7.754 occurrences and, consequently, its real frequency of occurrence probably does not exceed that which would give this number in our body of observations. Again, if n is 6, the 95 per cent point is 1.635, so that, if 3 cases have certainly been observed, the expectation probably exceeds 0.817, since for this value 95 per cent of the observed numbers will be 0, 1, or 2. We may thus use the table very simply to show just how much information about the frequency of rare events is contained in a record of only a few such occurrences.

6. *The probablity integral of "Student's" t distribution.*—It has already been shown that the ratio t of a deviation to its standard error as estimated from n degrees of freedom is

$$df = \frac{\dfrac{n-1}{2}!}{\dfrac{n-2}{2}!\sqrt{\pi n}}\left(1+\frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} dt;$$

or, if $\tan\theta$ is written for $t/\sqrt{n}$,

$$df = \frac{\dfrac{n-1}{2}!}{\dfrac{n-2}{2}!\sqrt{\pi}}\cos^{n-1}\theta \cdot d\theta.$$

Then the probability of exceeding a given value of t is

$$\int_{\alpha}^{\frac{1}{2}\pi} \frac{\dfrac{n-1}{2}!}{\dfrac{n-2}{2}!\sqrt{\pi}}\cos^{n-1}\theta \cdot d\theta,$$

where $t = \sqrt{n}\tan\alpha$.

Now, integrating by parts, it appears that

$$\int_\alpha^{\frac{1}{2}\pi} \frac{(r \pm \frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r+1}\theta \cdot d\theta = \left[\frac{(r+\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r}\theta \sin\theta \right]_\alpha^{\frac{1}{2}\pi}$$

$$+ \int_\alpha^{\frac{1}{2}\pi} 2 \frac{(r+\frac{1}{2})!}{(r-1)!\sqrt{\pi}} \cos^{2r-1}\theta \sin^2\theta d\theta.$$

which, so long as r is positive, is equal to

$$-\frac{(r+\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r}\alpha \sin\alpha + (2r+1) \int_\alpha^{\frac{1}{2}\pi} \frac{(r-\frac{1}{2})!}{(r-1)!\sqrt{\pi}} \cos^{2r-1}\theta \cdot d\theta$$

$$- 2r \int_\alpha^{\frac{1}{2}\pi} \frac{(r+\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{r+1}\theta \cdot d\theta.$$

Hence,

$$\int_\alpha^{\frac{1}{2}\pi} \frac{(r+\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r+1}\theta \cdot d\theta = -\frac{1}{2} \cdot \frac{(r-\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r}\alpha \sin\alpha$$

$$+ \int_\alpha^{\frac{1}{2}\pi} \frac{(r-\frac{1}{2})!}{(r-1)!\sqrt{\pi}} \cos^{2r-1}\theta \cdot d\theta$$

when r is positive; but, when $r = 0$,

$$\int_\alpha^{\frac{1}{2}\pi} \frac{(r+\frac{1}{2})!}{r!\sqrt{\pi}} \cos^{2r+1}\theta \cdot d\theta = \frac{1}{2}(1 - \sin\alpha).$$

Hence,

$$P = \frac{1}{2} - \frac{1}{2}\sin\alpha \left\{ 1 + \frac{1}{2}\cos\alpha + \frac{1\cdot3}{2\cdot4}\cos^4\alpha \cdots \right.$$

$$\left. + \frac{1\cdot3\cdot5 \cdots (n-3)}{2\cdot4\cdot6 \cdots (n-2)} \cos^{n-2}\alpha \right\},$$

when n is even. When n is odd, we proceed in the same way until $r = \frac{1}{2}$, and obtain

$$P = \frac{1}{2} - \frac{\alpha}{\pi} - \frac{\sin\alpha}{\pi} \left\{ \cos\alpha + \frac{2}{3}\cos^3\alpha \right.$$

$$\left. + \cdots \frac{2\cdot4 \cdots (n-3)}{3\cdot5 \cdots (n-2)} \cos^{n-2}\alpha \right\}.$$

As in the case of χ^2, when n is odd a transcendental function is required, in this case an inverse circular function, whereas when n is even, P is expressed as a function algebraic in t.

"Student" has given (1) four-figure tables of P up to $n = 20$; beyond this value a good asymptotic expansion is available (3). "Student's" tables are for $1 - P$ in the notation used above, and represent the probability of a value less than any given positive value of t. Since the distribution is symmetrical about zero, this probability is never less than $\frac{1}{2}$. For tests of significance, we often require the probability, $2P$, that the observed ratio of a deviation to its estimated standard error shall lie outside the limits $\pm t$, or the complementary probability, $1 - 2P$, that it shall lie within these limits.

It will help to make clear the analogy with the more general test of significance given by z, of which the χ^2 and t tests are special cases, to observe that, when n is even, the expansion for $1 - 2P$ is

$$\sin \alpha \left\{ 1 + \tfrac{1}{2} \cos^2 \alpha + \frac{1 \cdot 3}{2 \cdot 4} \cos^4 \alpha + \cdots \right.$$

$$\left. + \frac{1 \cdot 3 \cdot 5 \cdots n - 3}{2 \cdot 4 \cdot 6 \cdots n - 2} \cos^{n-2} \alpha \right\};$$

or, in terms of $\dfrac{n}{n + t^2} = \cos^2 \alpha = q$,

$$\sqrt{1 - q} \left\{ 1 + \tfrac{1}{2} q + \frac{1 \cdot 3}{2 \cdot 4} q^2 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots n - 3}{2 \cdot 4 \cdot 6 \cdots n - 2} q^{\frac{1}{2}(n-2)} \right\},$$

in which the expression within the bracket is the first $\frac{1}{2}n$ terms of the binomial expansion

$$(1 - q)^{-\frac{1}{2}}.$$

Just as the probability integral of χ^2 gives the partial sum of a Poisson series, so, therefore, does the probability integral of t give the partial sum of a special type of binomial expansion; in each case the external factor is the inverse of the sum of the complete series, and the identity holds for all even values of n.

7. *The probability integral of the distribution of z.*—The distribution of z involves two whole numbers, n_1 and n_2, which are the numbers of degrees of freedom in the two lines of the analysis of variance to be compared, and is given by the general formula,

$$df = 2 \frac{\dfrac{n_1 + n_2 - 2}{2}!}{\dfrac{n_1 - 2}{2}! \; \dfrac{n_2 - 2}{2}!} n_1^{\frac{1}{2}n_1} n_2^{\frac{1}{2}n_1} \frac{e^{n_1 z} dz}{(n_1 e^{2z} + n_2)^{\frac{1}{2}(n_1 + n_2)}}.$$

Writing

$$q = \frac{n_1 e^{2z}}{n_1 e^{2z} + n_2},$$

this becomes

$$df = \frac{\dfrac{n_1 + n_2 - 2}{2}!}{\dfrac{n_1 - 2}{2}!\,\dfrac{n_2 - 2}{2}!}\, q^{\frac{1}{2}(n_1-2)}(1 - q)^{\frac{1}{2}(n_2-2)}dq.$$

Now,

$$\int_q^1 \frac{(r + s + 1)!}{r!s!} x^r(1 - x)^s dx = \left[-\frac{(r + s + 1)!}{r!(s + 1)!} x^r(1 - x)^{s+1}\right]_q$$

$$+ \int_q^1 \frac{(r + s + 1)!}{(r - 1)!(s + 1)!} x^{r-1}(1 - x)^{s+1}dx$$

$$= \frac{(r + s + 1)!}{r!(r + 1)!} q^r(1 - q)^{s+1}$$

$$+ \int_q^1 \frac{(r + s + 1)!}{(r - 1)!(s + 1)!} x^{r-1}(1 - x)^s dx$$

$$- \frac{r}{s + 1}\int_q^1 \frac{(r + s + 1)!}{r!s!} x^r(1 - x)^s dx.$$

This establishes the recurrence relation

$$\int_q^1 \frac{(r + s + 1)!}{r!s!} x^r(1 - x)^s dx = \frac{(r + s)!}{r!s!} q^r(1 - q)^{s+1}$$

$$+ \int_q^1 \frac{(r + s)!}{(r - 1)!s!} x^{r-1}(1 - x)^s dx.$$

Hence, when n_1 is even, the probability of

$$\frac{n_1 e^{2z}}{n_1 e^{2z} + n_2}$$

exceeding any fractional value q is

$$P = (1 - q)^{\frac{1}{2}n_2}\left\{1 + \frac{n_2}{2} q + \frac{n_2(n_2 + 2)}{2\cdot 4} q^2\right.$$

$$\left.\cdots \frac{n_2(n_2 + 2)\cdots(n_2 + n_1 - 4)}{2\cdot 4 \cdots (n_1 - 2)} q^{\frac{1}{2}(n_1-2)}\right\}, \qquad (A)$$

of which the terms in the second component are the first $\frac{1}{2}n_1$ terms of the binomial expansion

$$(1 - q)^{-\frac{1}{2}n_2}.$$

From this expansion, when n_1 is even and not too large, the probability may be easily calculated.

Alternatively, using the direct result of integration by parts, we shall find the alternative expression

$$(1 - q)^{\frac{1}{2}(n_1 + n_2 - 2)} \left\{ 1 + \frac{n_1 + n_2 - 2}{2} \cdot \frac{q}{1 - q} \right.$$

$$+ \frac{(n_1 + n_2 - 2)(n_1 + n_2 - 4)}{2 \cdot 4} \left(\frac{q}{1 - q} \right)^2 + \cdots$$

$$\left. + \frac{(n_1 + n_2 - 2) \cdots (n_2 + 2)}{2 \cdot 4 \cdots (n_1 - 2)} \left(\frac{q}{1 - q} \right)^{\frac{1}{2}(n_1 - 2)} \right\} \tag{B}$$

involving the first $\frac{1}{2}n_1$ terms of the expansion of the positive binomial

$$\left(1 + \frac{q}{1 - q} \right)^{\frac{1}{2}(n_1 + n_2 - 2)}$$

The probability integral of z, when n_1 is even, is thus equivalent to the sum of $\frac{1}{2}n_1$ terms of a negative binomial in form (A), or of a positive binomial in form (B). It is of some historical interest that the probability integral of the normal distribution was first introduced by De Moivre as an approximation to the sum of a terminating series of binomial terms. Indeed, had the eighteenth century mathematicians possessed greater analytic power, the distribution of z, which was unknown to statisticians up to about 10 years ago, might have been studied before the normal distribution.

If n_2 tends to infinity and $n_2 q$ to the limiting value χ^2, both the forms (A) and (B) tend to the form

$$e^{-\frac{1}{2}\chi^2} \left\{ 1 + \frac{1}{2}\chi^2 + \frac{1}{2!} (\frac{1}{2}\chi^2)^2 + \cdots + \frac{1}{\frac{n_1 - 2}{2}!} (\frac{1}{2}\chi^2)^{\frac{1}{2}(n_1 - 2)} \right\},$$

which we obtained for the distribution of χ^2, if we identify n_1 of the general case with n of the χ^2 distribution. The distribution of χ^2 is, thus, as is obvious from its statistical derivation, the limiting case, when n_2 is infinite, of the general distribution, the substitution being

$$\frac{\chi'}{n} = e^{2z}, \qquad n = n_1.$$

Again, if $n_2 = 1$, the expression is evidently equivalent to that obtained for the probability that "Student's" test of significance t shall lie within the limits

$$\pm \sqrt{n_1 \frac{1-q}{q}},$$

but

$$\frac{1-q}{q} = \frac{n_2}{n_1} e^{-2z} = \frac{1}{n_1} e^{-2z},$$

hence the probability that z shall exceed a given value is the probability that t shall lie within the limits $\pm e^{-z}$, when $n_2 = 1$, $n_1 = n$.

Since z is the logarithm of the ratio of the estimates of the standard deviation derived respectively from n_1 and n_2 degrees of freedom, it follows that, if we interchange n_1 and n_2 and change the sign of z, the expression for the distribution is the same as before. Consequently, when n_2 is even, the probability integral may be expressed as the sum of $\frac{1}{2}n_2$ terms of a binomial expansion. The expression corresponding to (A) is, writing p for $1-q$,

$$1 - P = (1-p)^{\frac{1}{2}n_1} \left\{ 1 + \frac{n_1}{2} p + \frac{n_1(n_1+2)}{2\cdot 4} p^2 \right.$$
$$\left. + \cdots \frac{n_1(n_1+2) \cdots (n_1+n_2-4)}{2\cdot 4 \cdots (n_2-2)} p^{\frac{1}{2}(n_2-2)} \right\}, \quad (A')$$

illustrating that, for $n_1 = 1$, $n_2 = n$, the probability that z exceeds any given value is the probability that t will fall *outside* the limits $\pm e^z$, and that, when n_1 is infinite, the χ^2 distribution is given by the transformation

$$\frac{\chi^2}{n} = e^{-2z}, \qquad n = n_2.$$

Corresponding to expression (B), we have

$$1 - P = q^{\frac{1}{2}(n_1+n_2-2)} \left\{ 1 + \frac{n_1+n_2-2}{2} \frac{1-q}{q} \right.$$
$$\left. + \cdots \frac{(n_1+n_2-2) \cdots (n_1+2)}{2 \cdots (n_2-2)} \left(\frac{1-q}{q} \right)^{\frac{1}{2}(n_2-2)} \right\}. \quad (B')$$

When both n_1 and n_2 are even, it appears that P and $1-P$ are the first $\frac{1}{2}n_1$ terms and the remaining $\frac{1}{2}n_2$ terms of the expansion of

$$(p+q)^{\frac{1}{2}(n_1+n_2-2)},$$

where

$$\frac{q}{p} = \frac{n_1}{n_2} e^{2z},$$

the ratio of the sums of squares in the analysis of variances.

In cases where either n_1 or n_2 is even, the probability integral can be expressed as the incomplete sum of a binomial series, either with positive or negative exponent, the exponent being the half of an odd integer if either n_1 or n_2 is odd. The case remains to be considered in which both n_1 and n_2 are odd.

For this case we apply the recurrence solution as far as $r = \frac{1}{2}$, obtaining

$$P = \int_q^1 \frac{\dfrac{n_2 - 1}{2}!}{\dfrac{n_2 - 2}{2}!\sqrt{\pi}} \, x^{-\frac{1}{2}}(1 - x)^{\frac{1}{2}(n_1-2)}dx$$

$$+ 2\frac{\dfrac{n_2 - 1}{2}!}{\dfrac{n_2 - 2}{2}!\sqrt{\pi}}\,(1 - q)^{\frac{1}{2}n_1}q^{\frac{1}{2}}\left\{1 + \frac{n_2 + 1}{3}\,q + \cdots\right.$$

$$\left. + \frac{(n_2 + 1) \cdots (n_2 + n_1 - 4)}{3 \cdots (n_1 - 2)}\, q^{\frac{1}{2}(n_1-3)}\right\}.$$

The numerical coefficient of the second term, when n_2 is odd, is

$$\frac{2}{\pi}\,\frac{2 \cdot 4 \cdots (n_2 - 1)}{1 \cdot 3 \cdots (n_2 - 2)},$$

and the integral remaining at this stage is just double the one which has been already evaluated for the t distribution; so, putting

$$x = \sin^2 \theta, \qquad q = \sin^2 \alpha = \frac{t^2}{t^2 + n_2},$$

we find, since n_2 is odd,

$$P = 1 - \frac{2\alpha}{\pi} - \frac{2 \sin \alpha}{\pi}\left\{\cos \alpha + \tfrac{2}{3} \cos^3 \alpha + \cdots\right.$$

$$\left. + \frac{2 \cdot 4 \cdots (n_2 - 3)}{3 \cdot 5 \cdots (n_2 - 2)}\, \cos^{n_1-2}\alpha\right\}$$

$$+ \frac{2}{\pi} \cdot \frac{2 \cdot 4 \cdots (n_2 - 1)}{1 \cdot 3 \cdots (n_2 - 2)} \sin \alpha \cos^{n_1} \alpha \left\{ 1 + \frac{n_2 + 1}{3} \sin^2 \alpha + \cdots \right.$$

$$\left. + \frac{(n_2 + 1) \cdots (n_1 + n_2 - 4)}{3 \cdot 5 \cdots (n_1 - 2)} \sin^{n_1 - 3} \alpha \right\}$$

in terms of α, where α is connected with z by the equation

$$\tan \alpha = \sqrt{\frac{n_1}{n_2}} \cdot e^z.$$

Tables for z for the 5 per cent and the 1 per cent points of the distribution have been given for $n_1 = 1, 2, 3, 4, 5, 6, 8, 12, 24, \infty$; the last five values are in harmonic progression and enable the table to be interpolated in the manner which I have called asymptotic interpolation. For n_2, I have given values from 1 to 30, together with 60 and ∞; in this case again the series of values for 20, 30, 60, and ∞, may be used for asymptotic interpolation and the table thus gives four-figure values of z, an accuracy fully sufficient for all common purposes for all combinations of n_1 and n_2 except the region in which n_1 exceeds 24 and n_2 exceeds 30.

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ANNUAL SURVEY OF STATISTICAL INFORMATION:
CAPITAL FORMATION AND THE FLOW OF NATIONAL
INCOME IN THE UNITED STATES

By Charles F. Roos

Of recent years there has been much speculation and argument regarding the nature of savings and investments and the rates of income received by such groups as entrepreneurs, property owners, creditors, industrial managers, the white-collar class, workingmen, and farmers. Unfortunately, arguments and opinions on these questions have not generally been supported by facts, for the reason that adequate statistical information has not been available.

With the advent of the "New Deal" in the United States and the accompanying rapid growth of new governmental agencies, often created with the avowed intention of bringing about a "better" distribution of income and each possessing a division created to collect and interpret statistics, it was inevitable that a wealth of new data on the flow and use of income should become accessible. Furthermore, largely owing to the guidance of the new Central Statistical Board, composed of representatives of various departments and agencies, much of this information is thoroughly reliable. As a result of the activities of this Board, some of the older indexes have been revised to meet criticisms, and established departments of the government have often enlarged the scope of their work.

Such new organizations as the National Recovery Administration, which required statistics on a variety of economic problems, greatly stimulated the collection of data by both governmental and private agencies. During the life of NRA,[1] many of the six hundred or more code authorities which governed industries collected and analyzed statistics bearing on their particular activities. Data which had previously been gathered in inadequate samples by trade associations became vastly better because of increased coverage and many indexes were greatly improved.

Under NRA, some industries preferred to hire private statistical agencies, such as Dun and Bradstreet of New York City, Real Estate Analysts, Inc., of St. Louis, Missouri, and Price, Waterhouse and Company of New York, to collect and interpret their material. In spite of the Supreme Court decision, several of these organizations are continuing to publish statistical series begun at the instance of code authorities. Indeed, the *Real Estate Analyst* has so vastly enlarged its

[1] On May 27, 1935, the Supreme Court held the Act creating the organization to be unconstitutional.

statistical service, which began several years before the advent of
the New Deal for the purpose of rendering information on factors
affecting real estate, that its data now represent the most complete
and up-to-date on the subject. It is not, however, the only agency
which publishes new statistical information pertaining to building
and real estate: mention should be made also of the work of the Re-
search and Planning Division of NRA,[2] the Federal Home Loan Bank
Board, and the Federal Housing Administration.

I. STATISTICS PERTAINING TO BUILDING AND REAL ESTATE

In a study presented at the Philadelphia meeting of the Econometric
Society,[3] Roy Wenzlick, Victor S. von Szeliski, and the author, em-
ployed data for St. Louis covering the period 1900–34 to indicate
the inadequacy of the interest rate and the importance of the fore-
closure rate as a measure of availability of credit for financing new
building. The *Real Estate Analyst*, published by Real Estate Analysts,
Inc., under the direction of Roy Wenzlick, a member of the Econo-
metric Society, has recently been able to carry back a monthly index
of foreclosures for St. Louis to 1868. In addition, his organization now
publishes current weekly and monthly foreclosure figures for twenty-
three other metropolitan areas. Then, there are similar data collected
by the Federal Home Bank Board, which is now publishing the number
of real estate foreclosures by months from 1932 and by years from
1926, broken down by states and Federal Home Loan Bank Districts.
The figures are published for about a thousand identical communities
representing about 52 per cent of the population of the United States.
Both the *Real Estate Analyst* and the Federal Home Bank Board
base their statistics on reports received from recording officials of
counties, cities, townships, and other governmental agencies.

The Home Loan Bank Board and Real Estate Analysts also collect
information on real estate loans. The former publishes monthly data,
from December 1932, on the number and nature of assets of members,
applications for loans, loans advanced, repayments, balances out-
standing at the end of the month, lines of credit, and unused lines
of credit. The coverage is 100 per cent of the member lending cor-
porations. To supplement this, the *Real Estate Analyst* presents a series
of all new mortgages, federal and private, recorded in twenty-four
metropolitan areas by months from 1916 to the present.

In addition there are new accurate figures indicating rental return

[2] This is practically the only division of NRA which survived the Supreme
Court decision.

[3] C. F. Roos, "Factors Influencing Residential Building," *Dynamic Economics*,
A Monograph of the Cowles Commission for Research in Economics, Blooming-
ton, Indiana, 1934, Chapter VI, pp. 69–110.

to owners. Each month the *Real Estate Analyst* makes a vacancy estimate for St. Louis, based on a sample of 11,000 dwelling units. It occasionally publishes analyses of vacancy conditions in other cities, derived from special reports or data collected by the United States Bureau of Labor Statistics. Starting with the first week in January 1935, it has also presented weekly and monthly series for twenty-six principal cities showing the average advertised residential rent per month per room for both heated and unheated units. A similar series is given for St. Louis in quarterly and annual figures from 1850 to date and weekly and monthly since 1917. Again, the *Analyst* has published yearly figures on real estate taxes per family in Greater St. Louis for the period 1868 to date.

Both the *Real Estate Analyst* and the Home Loan Bank Board are now publishing fresh material on the number of new residential units. The former has a series giving the number of new family accommodations resulting from all new building in Greater St. Louis by months from 1886. In April 1935 the latter began to publish similar data for the United States as a whole, using reports made to the Bureau of Labor Statistics, which cover approximately 95 per cent of the population in all cities with 10,000 or more inhabitants.

Several new series on construction costs have recently become available. The *Real Estate Analyst* publishes data on costs for five different types of residential buildings for greater St. Louis, all items of material and labor having been computed quarterly sir..e September 1932 and yearly since 1913. One important feature of these series is that allowances are made for changes in style of fixtures, etc., so that costs are corrected for certain trends. Less exact cost figures are shown by the Federal Home Bank Board which publishes the cost of construction in hundreds of dollars as given by building permits for its bank districts and for states, monthly for all cities of 10,000 or over, beginning in 1933, monthly for all cities of 25,000 or over, starting in 1930, and annually for all cities of 25,000 or over, beginning in 1921. In using series based on permits issued, it should be remembered that many cities charge a permit fee based on the value of the building. As a result of the desire to avoid large permit fees, building cost as given by permits tends to be somewhat under actual costs as computed, for instance, by Wenzlick.

Finally, with respect to statistics pertaining to real estate, Wenzlick has also studied the relation of farm to urban values and has found certain interesting lags.

II. BANKING STATISTICS

The last two years have witnessed a notable improvement in statistical data on corporation financing through security issues and

also through bank loans. Important material showing the availability of credit for working capital loans during a depression period has also been obtained.

To determine the extent to which business men had difficulty in securing credit for working capital purposes, the Bureau of the Census undertook a survey suggested by the Small Industries Committee of the Business Advisory and Planning Council of the Department of Commerce. Among other startling findings was the fact that a great many business men reported that they had on hand large unfilled orders at profitable prices and were unable to obtain working capital loans to finance the production necessary to fill these orders. Even more amazing was the discovery that, of the establishments reporting difficulty in obtaining credit, 22.9 per cent fell within the very select group having current ratios (current assets to current liabilities) of 3.0 or better and over 90 per cent of these also had net worth to debt ratios of 3.0 and over. Still another interesting feature of the report was that between 1926 and 1934 fully three-quarters of small American businesses obtained working capital through bank loans, and at least 40 per cent used banks for long-term loans.

Further information on the nature of loans is now given by the Federal Reserve Board. Since 1917 the Board has published in the *Federal Reserve Bulletin* condensed weekly reports of principal assets and liabilities of banks in about 100 of the larger cities. During the latter part of 1934, the form of these reports was revised and enlarged to show considerably more information, especially with respect to the loans, discounts, and deposits, of the 350 or so currently reporting member banks in ninety-one leading cities. Another improvement was made by showing data on debits, or charges to deposit accounts at these banks. On the basis of this additional information, the Board's published statement was expanded in October 1934 to show separately, and by Federal Reserve districts, loans to brokers and dealers in New York and outside, loans on securities to others, acceptances and commercial paper bought, loans on real estate, and obligations fully guaranteed as to both principal and interest by the United States Government.

Since May 1930, the text accompanying the weekly Federal Reserve bank statement of condition has contained a brief table showing changes in the amount of Reserve bank credit outstanding and in the factors related thereto. In February 1934 these data were revised to 1917 to show separate figures for "Treasury and national bank currency" and "Treasury cash and deposits with Federal Reserve banks," which two items formerly were combined under "Treasury currency adjusted." Recently still further revisions have been made in the presentation of figures on monetary gold stock and money in

circulation. The series as revised will appear in Tables 1–6 of the 1934 *Annual Report of the Federal Reserve Board*, soon to be released. Furthermore, beginning with July 17, 1935, the weekly statement of condition of the Federal Reserve banks has included a preliminary estimate of excess reserves of member banks.

III. SECURITY AND EXCHANGE DATA

About the 15th of each month the Federal Securities and Exchange Commission releases statistics of registration statements declared effective by the organization. This material shows the types of securities registered with the Commission, estimated gross proceeds from their sale, amounts not presently offered for sale, costs of selling and distribution, channels of distribution, net proceeds, and contemplated use of funds raised by the issuers. The report segregates expenditures for new plants and equipment, purchase of investments, purchase of other assets, repayment of indebtedness, and working capital. Only grand totals for all securities declared effective are published, but a breakdown of this total by industries is kept by the Commission. In using these data it should be borne in mind that they do not include issues of the United States Treasury and other Federal agencies, states, municipalities, railroads, or charitable and educational institutions, nor do they include intrastate or various types of private placings, all of which are exempt from registration. The data cover intentions to sell securities as registered with the Commission, rather than securities actually sold.

The Securities and Exchange Commission also publishes monthly the volume and value of trading in stocks and bonds on each and all the registered exchanges. The figures include, in almost all cases, transactions which are not reported in the quotation sheets or on the ticker as, for instance, odd-lot transactions and stopped-stock on the New York Stock Exchange. In addition, the Commission is compiling a number of other series, some of which will not be published but will, however, be made available upon request, and some of which will be for the use of the Commission only.

During the past twenty-one months the Cowles Commission for Research in Economics of Colorado Springs has been preparing new Indexes of Investment Experience in Common Stocks, which it is expected may be completed and published within the next three to six months. These new series cover the movements of American stock prices by months from 1872 to 1918 (in some cases 1926) when they will be joined to the adequate Standard Statistics figures. The Cowles Commission Indexes are comprehensive in scope and embody all corrections for stock rights, stock dividends, and other capital changes.

The prices of the individual stocks comprising the indexes are properly weighted by shares outstanding. From the point of view of return on capital, certain features of the Cowles Indexes are worthy of note. Price statistics for particular industries have been extended over considerably longer periods than have hitherto been treated. In their first form, the indexes include the effect of cash dividend payments, the dividends being compounded back into the indexes. In their second form, the effect of cash dividends is excluded. New indexes of yields, with dividend rates weighted by dollar magnitude of individual disbursements, are to be given for different industries, and also for such larger classifications as public utilities and industrials.

Mention should also be made of the study of bond yields and interest rates which is being prepared by Frederick R. Macaulay of the National Bureau of Economic Research and is expected to be published shortly.

Such fresh findings as these will make possible a more exact knowledge of the return on capital as among different industries; they should also permit a closer analysis of the effect of disparities in yield in guiding the flow of funds to different phases of economic activity.

IV. GOVERNMENTAL EXPENDITURES, EMPLOYMENT, AND PAYROLLS

Data on Federal expenditures for relief and public works are published by several agencies. The Public Works Administration releases monthly data on loans to states, municipalities, and other subdivisions, and weekly expenditures and employment.

The Federal Emergency Relief Administration publishes each month a table showing the total amount of funds which the Administration has made available for emergency relief purposes to each state, and to the District of Columbia, and the territories of Alaska, Hawaii, Puerto Rico, and the Virgin Islands. Separate columns give the amounts allowed during the calendar months for which the report is issued and the cumulative amounts granted prior to the end of that month. Another table shows, by states, the amount of obligations incurred for emergency relief from Federal, state, and local public funds during the month. In separate columns are shown obligations incurred for (1) direct relief during the month, (2) earnings of relief persons, (3) non-relief persons, and (4) purchases of materials, supplies, and equipment. The table also gives total obligations incurred for the work program and for all other purposes.

The Federal Emergency Relief Administration also publishes a table showing the number of resident families and single persons receiving emergency relief from public funds. This table gives the number of resident single persons, the total number of resident cases, and the

total number of resident persons, who received aid at some time during
the calendar month for which the report is issued. Another table shows
the number of persons receiving each type of relief. Still other figures
indicate the number of cases which received advances under such
special programs as rural rehabilitation and college student aid. In
addition, there are tables giving (1) the number of resident cases re-
ceiving relief in each of approximately 146 urban areas, (2) the average
amount of relief extended per resident relief family for the principal
cities of each state, and for the remainder of the state, and (3) the
number of transients cared for under the Federal transient program.

The employment and payroll indexes of the Bureau of Labor Statis-
tics are being extended to include public employment and payrolls
for states, counties, and municipalities. The Bureau plans an index
which will show what portion of the national income goes to pay the
wages and salaries of employees of all units of government in the
country, and the number employed by governments. Data on these
matters, which have not been adequately covered previously, will
then be regularly tabulated. Extensions are also being made in the
Bureau's coverage of employment and payrolls in the distributing
trades and the construction industries. The Bureau's index in the field
of wholesale and retail distribution, which has been deficient because
of the use of too small a sample of firms in these lines of activity, will
be improved by increasing the coverage to the point where the index
will be fully representative.

V. COST OF LIVING AND FAMILY EXPENDITURES

The Bureau of Labor Statistics is in the process of reorganizing its
cost of living index. The current series, using weights calculated from
family expenditure studies made in 1919–20, is misleading because,
during the intervening period, significant changes have taken place in
spending habits. Automobiles, silk stockings, electric lights, and tele-
phones, have become essential parts of contemporary living.

For the past year the Bureau has been making elaborate family ex-
penditure studies for the purpose of securing modern weights for its
cost of living index. Twenty-two cities have already been covered
and plans have been made to extend the analysis over one hundred rep-
resentative communities throughout the country. The present budgetary
investigations of the Bureau throw light upon the allocations of wage
earners' and low salary workers' family incomes among various items,
and, in addition, show the purchasing habits of this portion of the
population in terms of the type of outlet used and seasonal variations
in consumption habits.

The Bureau proposes to publish the results of these family expendi-

ture investigations for each individual community. In fact, the data for several of these communities have already appeared in the *Monthly Labor Review*. In addition the Bureau expects to publish indexes based on regional differences and on size of communities. It is also contemplating construction of series for various types of economic community, such as the three kinds of areas, manufacturing, distributing, and agricultural.

In addition to revising its indexes of the cost of living, the Bureau is now engaged in establishing a retail price index, which will be weighted by the total sales of the various important commodities that enter the retail transactions of the country. This is especially significant because many commodities which do not appear in the budget of the average wage earner and low-salaried family at present find no place in any of the indexes.

The Bureau of Foreign Commerce of the United States Department of Commerce also now publishes several new series relating to the cost of living and retail sales. There are (1) a seasonally adjusted variety store sales index which shows variation in sales of identical stores from 1929 to date and represents 75 per cent of the total volume in the field, and (2) an index of rural general merchandise sales from 1929, based on mail sales of the three leading mail order houses and the sales of one major chain store operating mainly in rural areas. In this connection, it may be pointed out that the wholesale price index of the Bureau of Labor Statistics is also being thoroughly reorganized. Commodity specifications are being revised and a comprehensive study is being made of market areas where special types of prices prevail. The number of items included in the index is also being greatly increased.

Because of the importance of new passenger automobile sales as indicating a community's purchasing power[4] the index of such sales prepared by the Bureau of Foreign and Domestic Commerce must be noted. This is a seasonally adjusted index depicting dollar value of retail sales of all makes of cars. It is constructed by computing an average realized price of cars which, together with statistics of unit sales, is used in calculating the total dollar volume.

VI. FARM INCOME

Improvements have recently been made in figures showing income received by agricultural groups. For instance, revised index numbers

[4] Stephen M. DuBrul, *Analysis of the Automobile Market*, General Motors Corporation, Detroit, Michigan. See also the study by C. F. Roos and Victor Perlo, "Automotive Demand for Gasoline," Chapter iii, *Dynamic Economics*, *op. cit.*

of prices obtained by farmers for farm products were issued in 1934 by the Bureau of Agricultural Economics of the United States Department of Agriculture. The revision was begun in 1931 to make index numbers of farm prices more representative of the actual changes in the prices of all farm products by utilizing both the results of the 1930 Census and the additional data collected by the Crop Estimating Service. Besides including more commodities, such as dairy products and truck crops, the revised series is more representative of prices of all farm products in that the index numbers for each group of commodities are weighted in proportion to that group's contribution to total cash farm-income during the inter-census period from 1924 to 1929, whereas formerly the general series was computed from the weighted aggregate value for the 27 commodities used in the old series. The new series has been adjusted for seasonal variation by correcting each individual price series for normal seasonal changes before combining individual series into groups. However, index numbers with and without seasonal adjustments are released near the end of the month.

The Bureau of Agricultural Economics is now publishing monthly estimates of receipts from the sale of principal farm products and from Federal rental and benefit payments. These last two items are shown monthly by states since payments first began in August 1933. The figures for farm income, which date from January 1932, are based upon the cash income from 33 of the more important farm commodities from which farmers derive 94 per cent of all their income from the sale of products. The proportion of total farm marketings included in the various states ranges from 81 per cent in Massachusetts to 99 per cent in Nebraska. Although these series do not strictly portray total cash receipts of farmers by states, they do serve as an index of change in receipts in the various states, since they form such a large proportion of the total. The Bureau also publishes a new tabulation showing monthly cash income of farmers of the United States, based upon sales of 37 of the more important agricultural products. The data show income by groups of commodities, both in dollars and in index form, the index numbers being adjusted for seasonal variation and extending back to January 1924.

VII. CONCLUSION

From the foregoing descriptions it appears that the econometrist is confronted with a wealth of new statistical information pertaining to economic conditions in the United States. Just what this material will contribute to our knowledge of economics remains to be seen, but it appears that a veritable gold mine of fresh fact is now awaiting analysis. May the econometrist hasten to accept the challenge presented.

That there is reason to be optimistic in this regard is indicated by the fact that such analysis has already been started by several important research organizations. To give an instance, the Brookings Institution of Washington, D. C., has recently published works entitled (1) *America's Capacity to Produce*, (2) *America's Capacity to Consume*, and (3) *the Formation of Capital*.

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THE MARGINAL UTILITY OF MONEY IN THE UNITED STATES FROM 1917 TO 1921 AND FROM 1922 TO 1932

By Frederick V. Waugh

THE concept of utility is fundamental to the theory of value and also to other concepts such as real wages and standards of living. A rise in the standard of living, for example, means an increase in the degree of utility or satisfaction (the two terms being considered synonymous for the purposes of this discussion). Moreover, many principles of applied economics are based on theories of utility. An example is the principle of "equal sacrifice," or "equal disutility," among individuals as a criterion of taxation.

In spite of the obvious importance of this subject, economists until recently have made little progress in clarifying ideas about utility and in estimating the degree of utility derived by consumers from different goods and services at different periods. The recent theoretical researches of Irving Fisher[1] of Yale University and the theoretical and statistical researches of Ragnar Frisch[2] of the University of Norway have, however, made great advances in this field. They have shown that under certain conditions it is possible to measure quantitatively the variation in the marginal utility of money from one period of time to another or from one group of persons to another. It is quite possible that the development of these methods may give a new impetus to statistical price research and to a rapid development in quantitative economic science which will bring about a much needed coordination of statistical research and theory.

The writer became interested in the possibility of such research while studying under Professor Frisch during 1932–33,[3] and began at that time an attempt to adapt Frisch's methods to the study of the marginal utility of money in the United States as a whole. The results of Frisch's Paris and United States studies are commonly said to represent measurements of the marginal utility of money in France and in the United States, repectively. In the opinion of the writer these studies should be considered as measuring the marginal utility of money of selected groups of consumers and the results should not be applied to the whole populations of France and the United States.

[1] Irving Fisher, *Economic Essays contributed in honor of John Bates Clark*, Macmillan, 1927.

[2] Ragnar Frisch, "Sur un problème d'Économie pure," *Norsk Matematisk Forenings Skrifter*, Serie 1, Nr. 16, 1926 ... *New Methods of Measuring Marginal Utility*, Beitrage zur Ökonomischen Theorie, Tübingen, 1932.

[3] This was part of a year's study in Europe made possible by a fellowship of the Social Science Research Council.

The present study is intended to measure the marginal utility of money to the population as a whole. Such measures applied to a whole population (or to a group representing a typical cross-section of the whole population) are obviously important if the results are to be applied to such practical questions as the determination of fair rates of income taxes. It has been possible also in the present study to simplify the statistical procedure, although the method used is based directly on the method used by Frisch in his Paris study.

Method of measuring marginal utility.—The methods developed by Fisher and Frisch for measuring variations in marginal utility are based on the assumption that people spend their money in such a way that the marginal utility derived from a dollar's worth of each commodity or service purchased is equal to the marginal utility derived from a dollar's worth of each other commodity or service purchased. To the extent that consumers are rational and well informed, and to the extent that they are free to spend their money as they choose, this assumption must be correct. We will grant, of course, that as individuals we sometimes spend money unwisely, that is, we may impulsively buy things which give us less satisfaction than we could have obtained by spending the same amount of money for something else. Still, if we consider the purchases of a large group of persons, or of all the consumers in a country, most of these mistakes should cancel out, and the marginal dollar spent for shoes can be considered as giving the same utility as the marginal dollar spent for apples or for cigarettes.

Granting this assumption, we can proceed to demonstrate by the use of elementary mathematics the possibility of measuring changes in marginal utility.

Let us first adopt the following symbols:

U_a and U_a' represent the marginal utilities of commodity a in the periods 0 and 1, respectively.

U_b and U_b' represent the marginal utilities of commodity b in the periods 0 and 1, repectively.

P_a and P_a' represent the prices of commodity a in the periods 0 and 1, respectively.

P_b and P_b' represent the prices of commodity b in the periods 0 and 1, respectively.

Then from the first assumption we can state the relations:

(1)
$$\frac{U_a}{P_a} = \frac{U_b}{P_b},$$
$$\frac{U_a'}{P_a'} = \frac{U_b'}{P_b'},$$

which simply state that the marginal dollar (or cent, or franc, or other unit of money) spent for commodity a gives the same satisfaction as is obtained from the marginal dollar spent for commodity b.

We can compare the marginal utilities in the two periods by dividing one equation by the other, giving the one equation:

$$(2) \qquad \frac{U_a}{U_a'} \cdot \frac{P_a'}{P_a} = \frac{U_b}{U_b'} \cdot \frac{P_b'}{P_b}.$$

With the help of this equation we can determine the relative marginal utilities of commodity a in the two periods *if the marginal utility of commodity b was the same in the two periods.* For, if $U_b = U_b'$, the equation (2) becomes:

$$(3) \qquad \frac{U_a}{U_a'} = \frac{P_b'}{P_b} \cdot \frac{P_a}{P_a'} = \frac{P_a}{P_b} \left/ \frac{P_a'}{P_b'} \right. .$$

The condition that the marginal utility of b in the two periods must be equal is very important. We shall discuss further the possibility of finding cases where the condition is fulfilled.

First, however, let us establish two final equations which complete the statement of fundamental principles involved in the measurement of the marginal utility of money. We have considered a and b as being two commodities. If commodity a is money, we may conventionally put the price of a equal to unity, and express the price of b in relation to a. That is, we set $P_a = P_a' = 1$. Equation (3) then becomes (when a represents money):

$$(4) \qquad \frac{U_a}{U_a'} = \frac{1}{P_b} \left/ \frac{1}{P_b'} \right. .$$

In other words, the marginal utility of money at a given time compared with the marginal utility of money at another time is inversely proportional to the price of commodity b, if the condition is fulfilled that the marginal utility derived from b is the same in the two periods.

By means of equation (4) we can compare the marginal utility of the current dollar in two periods. It is also possible to compare the marginal utility of a unit of purchasing power in two periods. Let U_Σ represent the marginal utility of a complex consisting of all other goods and services except the commodity b. Let P_Σ represent an index of prices of the complex of goods and services, Σ. Then from equation (3) we get;

$$(5) \qquad \frac{U_\Sigma}{U_\Sigma'} = \frac{P_\Sigma}{P_b} \left/ \frac{P_\Sigma'}{P_b'} \right. .$$

This formula for measuring the relative marginal utility of a unit of purchasing power is similar to (4) for measuring the relative marginal utility of money, except that the price of b is "deflated" by dividing it by an index of prices of all other goods and services.

Principles of statistical application of the methods.—The above five equations cover the fundamental principles on which Frisch bases his methods of measuring marginal utility, although we have derived the equations in a way somewhat different from that used by Frisch. The validity of any statistical applications of these methods depends primarily on three things: first, on the accuracy of the first assumption that consumers distribute their purchases in such a way as to derive equal marginal utility from a dollar's worth of each commodity or service; second, on the accuracy and representativeness of the statistical data studied; and, third, on the finding of conditions under which the marginal utility of some commodity or service (or group of commodities or of services) is equal in the two periods to be compared. We may perhaps take for granted that the assumption as to the distribution of consumer expenditures is approximately true, at least of large groups of people. It may not always be strictly accurate even for a large group, since the consumer does not generally have entirely free choice but must spend a part of his income in such forms as the payment of direct taxes and interest on debts, and since we must grant that consumers make mistakes in judging the satisfaction they can get from goods or services. Yet, except for these qualifications, we may accept the assumption as being accurate when applied to large groups of consumers. The accuracy and representativeness of the statistical data and the condition that the marginal utility of commodity or service b must be equal in two periods need some further attention if statistical results are to be valid.

If we wish to measure the utility of money in a country as a whole, it is necessary to study a set of statistical data covering either the whole population, or covering a group chosen in such a way that it is a representative cross-section of the whole population. Such a sample group might be chosen either by some purely random method of sampling or by dividing the population into groups and sub-groups according to such factors as geographical location, nationality, income, and occupation, and choosing for study numbers of families in each group and sub-group in proportion to the total population.

Frisch's Paris study is based on data covering the purchases of members of a cooperative society, "l'Union des Cooperatures," from June, 1920, to December, 1922. Is it likely that this particular group of consumers derived from a given level of income the same utility as would an average group of Frenchmen? It seems to us unlikely,

although we have no way of forming a good judgment. Certainly a study of members of a consumers' cooperative in New York City would not be likely to give results which could be applied to the United States as a whole. Another point which needs to be considered is the period of time studied. The marginal utility of incomes, or even of "real" incomes, may vary a good deal from one period to another, particularly in periods of unsettled economic and financial conditions. From this standpoint it appears that the period chosen for the Paris study was probably about as satisfactory as any post-war period, in spite of the 1921 business depression and a fairly sharp drop in costs of living during this period. The period was before the sharp drop of the franc in foreign exchange and before the sharp rise in the cost of living in France which started toward the end of 1922.

The study made by Frisch in the United States was based on data from the budget study made by the Bureau of Labor Statistics in 1918–19.[4] The records studied covered nine of the ninety-two cities reported in the survey. Here, again, it is quite doubtful whether the families covered by the survey represented a good cross section of the population of the United States. The nine cities were all fairly large and no small towns nor villages were included. Moreover, in these cities there was a definite selection of certain kinds of families. For example, the requirements in the selection of families by the Bureau of Labor Statistics included: "1. *The family must be that of a wage earner or salaried worker, but not of a person in business for himself. The families taken should represent proportionally the wage earners and the low or medium salaried families of the locality*," and "5. *All items of income and expenditures of members other than those living as lodgers must be obtainable.*" Also the survey covered white families only. It seems very doubtful that we could assume that the utility of money to this particular group of consumers was typical of the population as a whole. Rather it should be considered as typical of wage earners and low or medium salaried white families living in large industrial cities and keeping some kind of record of their expenditures. The marginal utility which this group derives from its income may be quite different from that derived by the farmer, the shopkeeper, the doctor or lawyer, or others who do not work for wages or small salaries, or who live in small towns or villages, or who keep no records. In addition, the period was certainly not normal. Seventy-five percent of the records were taken in 1918, during the war, and the rest in 1919, immediately following the war. The war loan drives, the regulation of food purchases, and the rapidly rising levels of wages and prices during that period, can well be remem-

[4] U. S. Bureau of Labor Statistics, Bulletin No. 357, *Cost of Living in the United States*, May 1924.

bered, and such conditions undoubtedly affected the utility of money.

For the above reasons, it seems clear to the writer that the results of the Paris study and the United States study should be considered as applying only to the particular groups of families studied and mainly to the particular periods covered by the data. It would obviously be valuable to find a measure of the utility of money for the population of an entire country covering a considerable period of time. The simplest way to do this is to study data on incomes, expenditures, consumption, and prices, covering the entire population during a period of several years.

Preliminary studies.—In order to make such a study the writer gathered together the available statistical data on income in the United States and on the retail prices and consumption (or "statistical disappearance") of certain foods, including sugar, coffee, meat, and butter. With the kind assistance of Professor Frisch and his assistants at the Universitets Økonomiske Institutt at Oslo, these data were studied both by graphic and by mathematical-statistical methods. The results of some of these studies checked fairly well with those of Frisch's United States study, but in certain cases discrepancies were found which indicated the need for great care in selecting the commodity used as a standard of measurement. Further analysis of the data by Frisch has partially explained these discrepancies[5] but we believe that the problem of choosing the commodity continues to be very important.

Frisch's Paris study was based on the consumption and price of sugar, the utility of which was assumed to be practically independent of the utilities derived from other sources. In the preliminary work done by the writer in cooperation with Professor Frisch, we assumed that the utilities of coffee, meat, and butter, were also independent. Now, the assumption as to butter might be immediately questioned, since butter competes with oleomargarine, lard and lard substitutes, and probably with other foods. Yet there is no reason to think that the utility of coffee and meat (the meat statistics covered all beef, pork, and lamb, which is practically all the meat consumed) should be less independent of all other utilities than should sugar. We must grant that coffee probably competes to some extent with tea, and that meat competes to some extent with fish and eggs. Still, sugar in the United States also competes with corn syrup, molasses, maple syrup, and maple sugar, and with some other foods, and also the demand for sugar for canning purposes probably varies somewhat with the quantity of fruits available and with their prices.

It may be possible to modify our statistical technique in such a way

[5] Results of these studies have been published recently in Publication No. 5 of the Universitets Økonomiske Institutt, Oslo, Norway.

as partially to overcome the difficulties due to a substitution of commodities. However, our results would doubtless be much more accurate if we could find a commodity, or group of commodities, the utility of which is entirely independent of other utilities.

One other principle must be remembered in choosing a commodity for this purpose. We have discussed the need for studying a good cross section of the population. The commodity must be one which is generally used by all families. Thus, even if we could be certain that the utilities derived from snuff or limburger cheese were perfectly independent of other utilities, a study based on these commodities would measure only the utilities enjoyed by the consumers of snuff and of limburger cheese, unless those particular consumers could be assumed as typical of the whole population, an assumption which might be doubted.

Data used in present study.—In order to overcome all these difficulties as far as possible, we decided, first, to continue the analysis of consumption and price data covering the whole United States and, second, to study a group of commodities used by all families and a group whose utility can be assumed practically independent of all other utilities. We chose for this purpose data recently published[6] representing the total annual expenditures for all foods in the United States and the index of retail food prices. Perhaps there is no such thing as independent utility, in which case the whole attempt to measure marginal utility by this method is futile. The utility derived from food may depend somewhat on the kitchen stove and the dining room table. Yet it seems fairly safe to assume that from one year to the next the same population would get about the same satisfaction from the same quantities and kinds of food. Certain rather definite amounts and kinds of food are necessary to maintain life and health. There is no possible substitute.

A recent bulletin of the United States Department of Agriculture[7] gives four diets; a restricted diet for emergency use, an adequate diet at minimum cost, an adequate diet at moderate cost, and a liberal diet. Statistical data on the disappearance of food in the United States indicate that the average family consumes a diet between that described as adequate at minimum cost and as adequate at moderate cost. Such a diet of the average family is just about adequate to maintain life and health. We believe that the marginal utility of such a diet may be considered as being as nearly constant as the marginal utility of any

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[6] Robert R. Doane, *The Measurement of American Wealth*, New York and London, Harper & Bros., 1933.

[7] Hazel K. Stiebling and Medora M. Ward, *Diets at Four Levels of Nutritive Content and Cost.* U. S. Department of Agriculture Circular No. 296, November, 1933.

group of commodities or services and as much more nearly constant than the utility of a given quantity of any single commodity or service.

By taking the whole group of foods as a basis for measuring utility, we also simplify the statistical problem, because the consumption in the United States of all foods together has been almost constant during periods of several years and is practically unaffected even by severe business depressions or great prosperity. The reason is that agricultural production as a whole responds only very slowly to changes in prices.
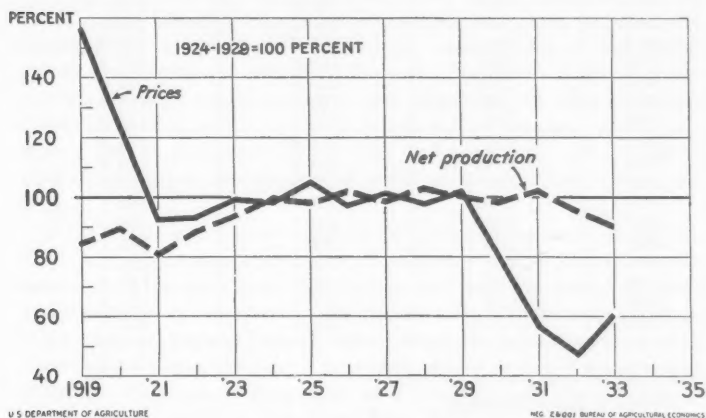
FIGURE 1.—Indexes of Net Agricultural Production and
Prices, United States, 1919–1933.

Figure 1 shows indexes of net agricultural production (that is, production for sale) and farm prices from 1919 to 1933. It is apparent that net production was practically constant from 1923 to 1932 in spite of the sharp drop in prices from 1930 to 1932. In 1933 production was reduced by adverse weather conditions and by production control operations of the Agricultural Adjustment Administration. The index in Figure 1 is based on all agricultural production for sale, rather than on the production of foods alone, but if it were modified to exclude non-food products such as cotton, wool, and tobacco, it would indicate the same conclusions.

Further evidence on this point covering a longer period is available in the statistical data we shall use in the following analysis. Table 1 and Figure 2 show the per capita expenditure for foods in the United States as estimated by Doane, compared with the index of food prices of the United States Bureau of Labor Statistics.

TABLE 1.—PER CAPITA EXPENDITURE FOR FOOD AND INDEX OF
FOOD PRICES, 1913 TO 1932

Year	Per capita food expenditures		Index of food prices[b] 1913 = 100
	Actual[a]	Index 1913 = 100	
	Dollars		
1913	103	100.0	100.0
1914	105	101.9	102.4
1915	107	103.9	101.3
1916	129	125.2	113.7
1917	160	155.3	146.4
1918	181	175.7	168.3
1919	194	188.3	185.9
1920	210	203.9	203.4
1921	158	153.4	153.3
1922	165	160.2	141.6
1923	177	171.8	146.2
1924	174	168.9	145.9
1925	180	174.8	157.4
1926	189	183.5	160.6
1927	186	180.6	155.4
1928	194	188.3	154.3
1929	191	185.4	156.7
1930	167	162.1	147.1
1931	152	147.6	121.3
1932	133	129.1	102.1

[a] Calculated from Doane, *op. cit.*
[b] Bureau of Labor Statistics.
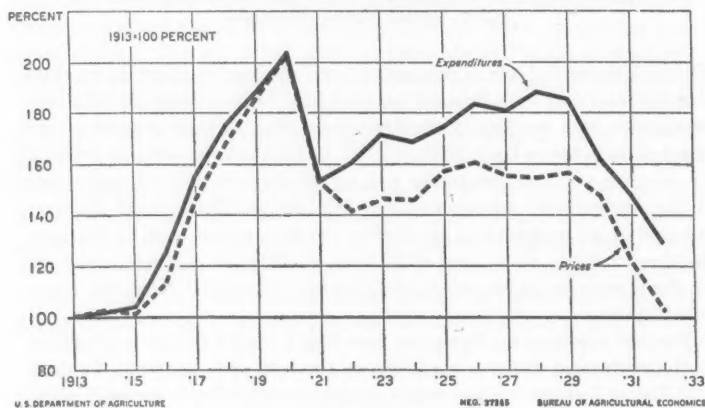
FIGURE 2.—Indexes of Food Prices and of Per Capita
Expenditures for Food, 1913–1932.

A comparison between the estimates of per capita food expenditures and food prices shows that in the period 1916 to 1921 there was almost perfect correlation between the two series and that the index of expenditures was slightly higher than the index of prices. Since expenditures increased more than prices from 1913 to 1916, we can assume that there was some increase in per capita consumption or some shift in consumption to the higher priced foods. In the period from 1922 to 1932 there was again almost perfect correlation between per capita food expenditures and food prices, at a level higher than that of 1916 to 1921, and this indicates either a substantial increase in per capita consumption or else a shift toward the consumption of the higher priced foods from one period to the next. Moreover, if the data are correct, the shift was abrupt and took place between 1921 and 1922. Considering each of the two periods separately, we come to the conclusion that *in the period 1916 to 1921 and again in the period 1922 to 1932 the per capita consumption of foods was almost constant from year to year*.

If the above conclusion is correct our problem is greatly simplified. That is, if the average diet in a given period was constant, we can assume that the marginal utility of that diet was very nearly constant and thus we have a convenient and reliable base with which to compare other utilities. To use Frisch's terminology, we have here a series of observations along the same "isoquant" which makes it unnecessary to interpolate a set of "isoquants" as Frisch did in his Paris study.

We should, perhaps, point out here that any such estimates as Doane's figures on food expenditures are only approximations, and that the degree of error can not be accurately determined.[8] However, they are based on a consideration of all available statistics of production, marketings, imports, exports, and stocks of food commodities, and their prices. The accuracy of the index of food prices might also be questioned, since it does not cover all foods in all places, but 43 foods in 51 cities. We must recognize that practically all statistics of prices and quantities of commodities produced or consumed are approximations. Our job is to get the most reliable data available and to use them remembering their limitations. Thus, we can not be sure that our base—the marginal utility of per capita food consumption—was

[8] Since this paper was written, two other studies of expenditures in the United States have been published. While there is considerable variation in the three available estimates, they all give practically the same results when used to measure changes in the marginal utility of money. The two recent publications are: Clark Warburton, "How the National Income was Spent, 1919–29," *Journal of the American Statistical Association*, Vol. 30, No. 189A, March, 1935, and William Henry Lough, *High-Level Consumption*, McGraw Hill Company, New York City, 1935.

exactly fixed at one level from 1916 to 1921 and at another level from 1922 to 1932, although the best available statistical evidence indicates that it was almost constant in each period and we shall assume as a first approximation, at least, that it was actually constant. The validity of the approximation will be supported by further analysis.

The increase in consumption from 1916–1921 to 1922–1932 can be explained by the fact that the production of foods in the United States was increased during and immediately following the war and that after the war the foreign outlet for foods was greatly reduced, leaving more food to be eaten by domestic consumers. Figure 1 shows a decided increase in net agricultural production from 1921 to 1924, after which it was almost the same from year to year.

Estimated marginal utility of the current dollar.—Year-to-year changes in the marginal utility of the dollar spent for all things except food can be estimated in each of the two periods, 1916 to 1921 and 1922 to 1932, by substituting in equation (4) the index of food prices in any two years in the same period. If one of the years in each period is chosen as a base and given the value 100, then the estimated relative marginal utility in each of the other years in the same period can be estimated by substituting in (4) the index of food prices in each of the other years.

Table 2 gives the estimated relative marginal utility of the dollar in each of the two periods as measured by (4). The marginal utility of the dollar in 1917, for example, relative to the marginal utility of the dollar in 1916 is measured simply by dividing the food price index in 1916 by that in 1917 and multiplying by 100. That is, the estimated relative marginal utility of money in 1917 is $113.7 \times 100/146.4 = 77.7$, or 77.7 percent of the marginal utility of money in 1916.

It is impossible to compare the marginal utility of the dollar in a year in the first period with that in a year in the second period. Each period must be considered separately. However, it is possible to estimate the relative, or comparative, marginal utility of the dollar in any two years in the same period. In Table 2 we have chosen the first year in each period as a base, and estimated the marginal utility of the dollar in each other year in the period as a percentage of the base year.

The results in Table 2 give a measure of the variation of the marginal utility of money with time. We shall subsequently consider the variation in the marginal utility of money in relation to the variation in purchasing power of money and in relation to the variation in income.

We have already remarked that such estimates should be considered as first approximations. Before we can accept them as being somewhere nearly right, we should have a good logical and statistical explanation for the variations in the estimates from year to year. Why did the marginal utility of the dollar drop from 1916 to 1920 and rise in 1921?

TABLE 2.—ESTIMATED RELATIVE MARGINAL UTILITY OF DOLLARS SPENT
FOR ALL THINGS OTHER THAN FOOD

(On the assumption that the consumption of food per capita was constant from
1916 to 1921; and also constant from 1922 to 1932)

Year	Food price index (1913 = 100)	Estimated relative marginal utility of dollars[a] (1916 = 100)
(a) From 1916 to 1921		
1916	113.7	100.0
1917	146.4	77.7
1918	168.3	67.6
1919	185.9	61.2
1920	203.4	55.9
1921	153.3	74.2
	(1913 = 100)	(1922 = 100)
(b) From 1922 to 1932		
1922	141.6	100.0
1923	146.2	96.9
1924	145.9	97.1
1925	157.4	90.0
1926	160.6	88.2
1927	155.4	91.1
1928	154.2	91.8
1929	156.7	90.4
1930	147.1	96.3
1931	121.3	116.7
1932	102.1	138.7

[a] The estimates are obtained from equation (4) using the food price index as P_b. The estimate for any year, i, in relation to the base year, 0, can be calculated as follows: $_iU = \dfrac{_0P_b \times 100}{_iP_b}$.

Why did it it rise from 1929 to 1932? We may expect the marginal utility of the dollar to be influenced by several kinds of forces. One of these forces is the purchasing power of the dollar. Obviously, if other things are equal, a dollar which has a high purchasing power is of greater utility than one with low purchasing power. A second factor is the income of consumers, or the number of dollars they have to spend. If consumers have low incomes and, therefore, can purchase only small amounts of goods and services, the marginal value of a dollar will be greater to them than if they had plenty of money. For example, when a man has an income of ten dollars a week, an increase to eleven dollars a week would represent a decided increase in utility, while if he had an income of 100 dollars a week, an increase to 101 dollars a week would add much less utility. Several other factors may cause variations in the utility of money, and these factors merit careful study. The reason

their influences have not yet been studied statistically is that there have been no measures of marginal utility of money. Some of these factors are: sudden changes in the price level or in income, which may cause the utility of money to rise or fall temporarily from the level ordinarily expected; the control of consumption or spending by governmental regulation or by public opinion in the time of war or other emergency; unsettled monetary conditions leading either to hoarding money in anticipation of increases in its value or to unusual spending in anticipation of a rise in prices; and changes in the distribution of income among the population.
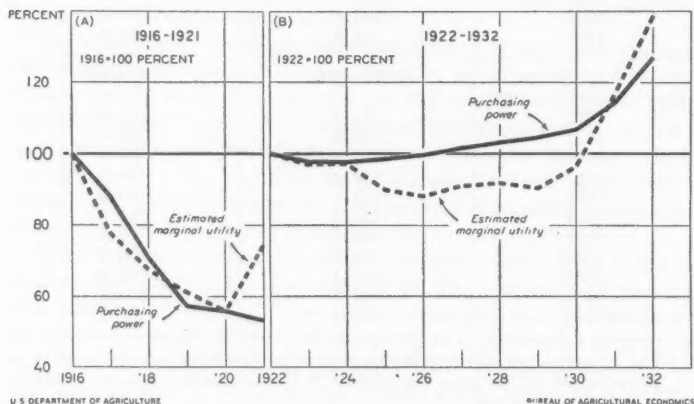
FIGURE 3.—Purchasing Power and Estimated Marginal Utility of the Non-Food Dollar, 1916–21 and 1922–32.

We shall not attempt in this particular study an analysis of the influence of all these factors on the utility of money but shall show that our first approximations in Table 2 are closely related to the buying power of the dollar and to consumer incomes. Some of the other factors mentioned above doubtless also influenced money utility, but our data are not adequate as a basis for analyzing their effects. Figure 3 compares the estimates of the marginal utility of the dollar spent for everything else but food (as given in Table 2) with the purchasing power of the non-food dollar, that is, with the reciprocal of the index of cost of living, excluding food. The Bureau of Labor Statistics index was used for this purpose and recalculated to exclude food prices. The annual indexes are simple averages of the indexes during each current year and the index for December of the previous year.

The estimates of the marginal utility of the non-food dollar are

evidently positively correlated with the buying power of the non-food dollar, as we should expect. This fact alone should not lead us to place too great reliance on the estimates of marginal utility, however, because

TABLE 3.—ESTIMATED RELATIVE MARGINAL UTILITY OF "REAL" DOLLARS, (OR OF PURCHASING POWER)

(Based on the assumption that the consumption of food per capita was constant from 1916 to 1921; and constant from 1922 to 1932)

Year	Index of cost of living excluding food $1913 = 100$ P_z	Index of food prices $1913 = 100$ P_b	$\dfrac{P_z}{P_b}$	Relative Marginal utility of real dollars[a] $1916 = 100$
(a) From 1916 to 1921				
1916	111.6	113.7	98.2	100.0
1917	126.7	146.4	86.5	88.1
1918	157.6	168.3	93.6	95.3
1919	195.2	185.9	105.0	106.9
1920	231.2	203.4	113.7	115.8
1921	209.2	153.3	136.5	139.0
	$1913 = 100$	$1913 = 100$		$1922 = 100$
(b) From 1922 to 1932				
1922	186.0	141.6	131.4	100.0
1923	190.4	146.2	130.2	99.1
1924	190.6	145.9	130.6	99.4
1925	188.8	157.4	119.9	86.7
1926	186.7	160.6	116.3	88.5
1927	183.3	155.4	118.0	89.8
1928	180.2	154.3	116.8	88.9
1929	177.9	156.7	113.5	86.4
1930	174.0	147.1	118.3	90.0
1931	162.6	121.3	134.0	102.0
1932	146.6	102.1	143.6	109.3

[a] These estimates are based on equation (5) where the marginal utility in 1916 and in 1922 are arbitrarily given the value 100. The marginal utility of real income in any year, i, relative to that in the year 0, is measured by

$$_iU = 100 \; \frac{(_iP_z}{(_iP_b} \bigg/ \frac{_0P_z)}{_0P_b)}.$$

it is apparent that a correlation between food prices and prices of non-foods would lead to this result, since our estimate of the marginal utility of the non-food dollar is based on an index of food prices. We are interested here not so much in the correlation between the two series as in their differences. Figure 3 indicates that from 1916 to 1917 and 1918 the marginal utility of the dollar dropped more than its purchasing power; in 1919 and 1920 it dropped less than its purchasing

power; and in 1921 it rose more than its purchasing power. It also indicates that after 1924 the marginal utility of the non-food dollar dropped to a lower level, where it remained for six years in spite of an increase in purchasing power, and that after 1929 the marginal utility of the non-food dollar rose much more than did its purchasing power.

We shall next show that these tentative conclusions are logical and that the differences between the estimated marginal utility and the purchasing power of the non-food dollar are largely explained by variations in real income of consumers.
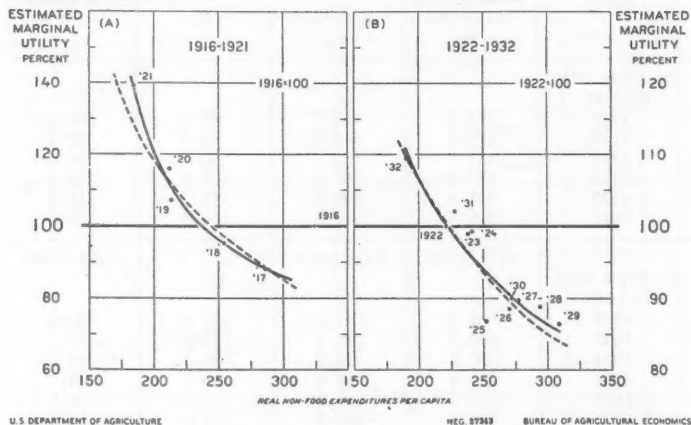
FIGURE 4.—Real Non-Food Expenditures Per Capita and Estimated Marginal Utility of Non-Food Expenditures of U. S. Consumers, 1916–1921 and 1922–1932.

The Marginal Utility of "Real" Dollars, or Dollars of Constant Purchasing Power

By the marginal utility of real dollars we mean the marginal utility of a given complex of goods and services. A real dollar, or a unit of purchasing power, will buy the same quantity of this complex of goods and services at any time. The marginal utility of real dollars excluding the amount spent for food can be measured by equation (5). Estimates based on equation (5) are shown in Table 3.

There is a high negative correlation between the year-to-year variations in the estimated marginal utility of real dollars, spent for other than food, in each of the two periods and the per capital real income from year to year. This relationship is shown in Table 4 and in Figure 4.

TABLE 4.—PER CAPITA REAL INCOME IN THE UNITED STATES AND ITS
ESTIMATED RELATIVE MARGINAL UTILITY; 1916 TO 1921 AND 1922 TO 1932

Year	Non-food expenditures per capita[a] (current dollars)	Index of cost of living excluding food[b] 1913 = 100	Real per capita non-food expenditures (1913 dollars)	Estimated relative marginal utility of real non-food expenditures[c] 1916 = 100
(a) 1916 to 1921				
1916	361	111.6	324	100.0
1917	362	126.7	286	88.1
1918	400	157.6	254	95.3
1919	416	195.2	213	106.9
1920	489	231.2	212	115.8
1921	385	209.2	184	139.0
				1922 = 100
(b) 1922 to 1932				
1922	411	186.0	221	100.0
1923	454	190.4	238	99.1
1924	459	190.6	241	99.4
1925	476	188.8	252	86.7
1926	504	186.7	270	88.5
1927	508	183.3	277	89.8
1928	529	180.2	294	88.9
1929	549	177.9	309	86.4
1930	473	174.0	272	90.0
1931	371	162.6	228	102.0
1932	278	146.6	190	109.3

[a] Calculated from Doane, *op. cit.*
[b] Calculated from Bureau of Labor Statistics Indexes eliminating the food group and averaging all reports during each year and the report of the previous December.
[c] From Table 3.

There appears to have been an almost perfect relationship between real per capita non-food expenditures and the estimated marginal utility of purchasing power in the years 1917 to 1921 and a fairly high relationship in the period 1922 to 1932. The estimated marginal utility in 1916 does not seem to line up with those for the years 1917 to 1921. This need cause us no concern since there are obvious grounds for considering the war and immediate post-war period as being significantly different from pre-war years. During and immediately after the war, the marginal utility of money was undoubtedly influenced by food regulations, Liberty Bond campaigns, and other events connected with the war.

We have drawn through the data in Figure 4 two smooth lines which may be considered as two estimates of the relationship of the marginal utility of purchasing power to real per capita expenditures (or, ap-

proximately, the relation of the marginal utility of money to real income). One of these lines was drawn free-hand through the data as plotted on the chart and the other represents a particular form of mathematical function which will be discussed below. It will be of some interest to note the general characteristics of the observed relationships as determined by the data themselves. We note (1) that the relationship is negative, i.e., that as real incomes increase the marginal utility of real dollars decreases; (2) that the curves are concave, indicating that the rate of change in marginal utility becomes less as expenditures increase; and (3) that the curve in the first period is much steeper than in the second. (The marginal utility scale used in plotting the data of the first period is one-half the scale used for the data of the second period. If both periods were plotted on the same scale the difference in slope would be apparent).

The shape of curve representing marginal utility of money with respect to real income.—The shape of the curve of marginal utility of money in relation to real income is very important in economic theory. Many economists have speculated about the probable shape of such a curve. Although there is a general agreement among economists that the marginal utility of money decreases as real incomes increase, there are important differences concerning other characteristics of the curve.[9] These differences are largely differences in assumptions based on logical considerations. Statistical studies such as the present one and those of Frisch give us the opportunity of using market price data as a check on such logical considerations in order to get a closer approximation to the true shape of the curve. This is definitely a problem of econometrics.

From a purely logical basis Frisch has stated five conditions which describe the probable shape of a curve representing the relation of the marginal utility of money to income. These conditions are:

(I) There exists a positive number, A (the minimum of existence), such that $f(R) > 0$ and has first and second derivatives for $A < R < \infty$.

That is, a certain positive income can be assumed to be necessary for existence. An income of this amount would give positive utility and if income were increased from this point the utility derived from it would be a function of the amount of income. This is clearly true.

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[9] The major differences will be found by comparing the reasoning and the mathematical formulas in the following studies: Daniel Bernouilli, *Specimen Theoriae novae de Mensura Sortis, Comentarii acadamiae scientiarum imperialis Petropolitanae* v, 1738.

Ragnar Frisch, "Sur un Problème," ... *op. cit.*

Umberto Ricci, "Die Kurve des Geldnutzens und die Theorie des Sparens," *Zeitschrift fur National Ökonomie*, III, Heft 3, 1932.

(II) $\underset{R \to A}{\text{Lim}} f(R) = \infty$

$\underset{R \to \infty}{\text{Lim}} f(R) = 0.$

That is, Frisch maintains, first, that if income is reduced to approximately the minimum of existence the marginal utility of money will approach infinity and, second, that if income is increased indefinitely the marginal utility of money will decrease but will never become zero, nor will it have a minus utility.

Both of these conditions—particularly the second—have been criticized recently by Ricci.[10] Ricci admits that the marginal utility of money in the neighborhood of the minimum of existence is doubtless very high and that it may be infinite, but he suggests we can not be sure of such an assumption. It seems to us that it would really be infinite. For example, if a person had exactly enough food to keep him alive, can we not assume that that food would be of infinite value to him?

The other condition is more seriously disputed by Ricci who maintains that a static curve of marginal utility of money would actually reach zero (and perhaps would have minus values), if income were indefinitely increased. Ricci would define a static curve as one representing a person's opinion of the utility he would get from various incomes. In reasoning about the shape of such a curve he says that, although men—at least in Europe and America—really have wants which are limitless, still any individual receiving a small income will believe that beyond a certain rather definite limit further increases in income would be useless to him. But, says Ricci, if this individual's income were increased until it reached or passed this imagined limit, his wants would increase and he would never actually reach the point when he was completely indifferent whether he received one dollar more or less than he expected.

Ricci would maintain that the relation between the actual incomes received and the actual enjoyment derived from them is a dynamic relationship and that the only true static relationship is the one imagined by the individual. We should consider both as being static and we are much more interested in the actual relationship than in the imaginary one. It seems to us that a dynamic relationship in economics should be defined as one which indicates how one variable at the time (t) depends on other variables at previous times ($t-1$), ($t-2$), etc. Thus, the marginal utility of money may depend not only on this year's income but also on whether income is increasing or decreasing. This would be a dynamic relationship. But the relationship of amounts

[10] Umberto Ricci, *op. cit.*

of income actually received this year and the enjoyment actually derived from it this year is clearly a static one.

Moreover, we are not satisfied with Ricci's assumption that each individual would imagine that beyond a certain limit he would derive no further utility from increases in income. In order to satisfy our curiosity on this point we asked a number of friends if they believed there would be such a limit in their individual cases. About one-half of them thought there would be and the other half thought not. *The combined marginal utility of money for the whole group would, therefore, never reach zero although this would be a static curve based on the judgment of individuals.*

(III) $\dfrac{df(R)}{dR} < 0$ between the limits $A < R < \infty$.

That is, the marginal utility of money decreases with any increase in income beyond the minimum of existence. This point is commonly recognized by all students of money utility.

(IV) The elasticity, or flexibility, of money utility with respect to income, $\dfrac{df(R)}{d(R)} \cdot \dfrac{R}{f(R)}$, is greater than unity for small values of $R(>A)$.

For example, imagine an individual with an income barely enough to sustain life. The marginal utility of money would be very high. If income were increased, say, 10 percent, the marginal utility of money would probably drop much more than 10 percent.

(V) $\underset{R \to \infty}{\text{Lim}} \dfrac{df(R)}{d(R)} \cdot \dfrac{R}{f(R)} = 0$.

That is, if income were increased indefinitely the flexibility of money utility with respect to income would approach zero as a limit. This condition could be illustrated as follows. Imagine an individual receiving a small income. If his income were doubled the marginal utility of money to him would be reduced. If it were quadrupled it would be reduced further. If it were multiplied by eight it would be still further reduced. If the process were continued indefinitely, a point would be reached where the utility added by doubling the income again would become very small. It would never quite reach zero, however, as can be seen from condition (2).

We believe that all five of Frisch's conditions are logical. The next question is: does an examination of the data support these theoretical conclusions?

We can throw light on this question by fitting different types of curves to the data. The equations suggested by Bernouilli, Frisch, and

Ricci, may be stated (where U represents the marginal utility of money and K, A, and C, are positive constants):

(6) Bernouilli's equation: $U = \dfrac{K}{R - A}$

(7) Frisch's equation: $U = \dfrac{K}{\log R - \log A}$

(8) Ricci's equation: $U = \dfrac{K}{\sqrt{R - A}} - C$.

Bernouilli's equation, (6), meets the first four of Frisch's conditions but the *flexibility* of marginal money utility as given by Bernouilli's formula is $\dfrac{-R}{R - A}$, a value which is greater than unity for all incomes. Such flexibility over a large range of incomes appears unlikely.

Frisch's equation, (7), meets all five conditions.

Ricci's equation, (8), does not meet condition (II) since, when incomes are increased high enough, the formula gives a negative marginal utility. It also does not meet condition (V), since the curve is elastic for very small incomes, becomes inelastic for moderate incomes, and again elastic for large incomes, until a "saturation point" is reached where the curve is infinitely elastic.[11]

It would be difficult to fit Ricci's curve to the statistical data. However, the results of the present study, of the studies made by the writer in Oslo, and of Frisch's studies in Paris and in the United States, all indicate that the curve actually becomes less and less elastic as incomes increase, within the range of the observed data. These results are not due to the mathematical form of curves chosen to fit the data. The same results have been obtained consistently by graphic analysis of all sets of statistical data studied.

Bernouilli's equation, (6), and Frisch's equation, (7), both result in curves which continue to become less elastic as incomes increase and for that reason fit the observed data more satisfactorily than does Ricci's equation. In Figure 5 we have attempted to discover which of these equations best fits the observed data. We have also included another possible equation which satisfies all five conditions as stated by Frisch and discussed above. That equation is:

(9) $$U = \dfrac{K}{\log \log R - \log \log A}.$$

[11] This is in opposition to Ricci's statement on page 323 of the article cited above. Ricci says the curve is inelastic in a first arc, elastic in a second, and again inelastic in a third.

The flexibility of the marginal utility of money according to this equation is

$$(10) \qquad F = \frac{df(R)}{dR} \cdot \frac{R}{f(R)} = \frac{-(\log_{10} e)^2}{\log R(\log \log R - \log \log A)},$$

which shows that equation (9) as well as equation (7) indicates very elastic curves for small incomes and that the elasticity diminishes and approaches zero as incomes are indefinitely increased.

It can be shown that similar equations involving such functions as log log log R also satisfy the above five conditions. Thus, it is possible to fit to the data a series of mathematical curves, all of which are equally logical, and to choose the one which best fits the data.[12]
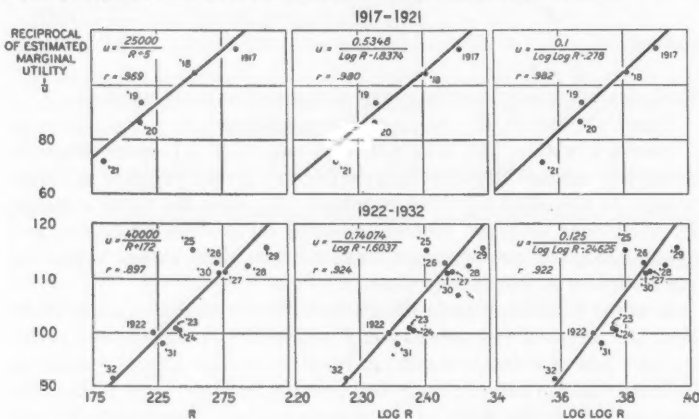
FIGURE 5.—Scatters of Reciprocal of Estimated Marginal Utility of Real Non-Food Expenditures and Certain Functions of Real Non-Food Expenditures, (R), 1917–1921 and 1922–1932.

In Figure 5 are plotted the scatter diagrams of the reciprocal of estimated marginal utility of real dollars ($1/u$) and R, log R, and log

[12] A similar choice of curves is usually possible and desirable in studying economic data. We believe, for example, that it is a serious mistake to choose on *a priori* grounds a particular mathematical equation for a demand curve and to proceed to fit that equation to all kinds of data without inquiring whether or not some other equally logical equation may fit the data better. Graphic smoothing of the data is usually preferable to such mechanical curve fitting. However, it is of interest to compare the results of graphic smoothing with the results obtained from fitting definite mathematical functions in the attempt to determine the nature of underlying relationships. In doing this it should be remembered that it is almost always dangerous to extrapolate any form of curve beyond the range of the data.

log R. These were plotted in order to determine which of equations
(6), (7), and (9), fitted the data most closely. This will be shown by the
scatter which is clustered most closely around a straight line. It is
apparent that the scatter in $(1/u, R)$ is less nearly linear in both periods
than are the other two scatters. There is little choice between the scat-
ters of $(1/u, \log R)$ and $(1/u, \log \log R)$ in either period. Both fit very
well in the first period and fairly well in the second. The scatter in
$(1/u, \log \log R)$ is slightly more linear in the first period and the scatter
in $(1/u, \log R)$ is slightly more linear in the second, but in both periods
it is doubtful if the "goodness of fit" of the two equations is significantly
different. The correlation coefficients between $1/u$ and $f(R)$ are:

a. Period 1917 to 1921
   Between $1/u$ and R; $r = +.969$
   Between $1/u$ and $\log R$; $r = +.980$
   Between $1/u$ and $\log \log R$; $r = +.982$
b. Period 1922 to 1932
   Between $1/u$ and R; $r = +.897$
   Between $1/u$ and $\log R$; $r = +.924$
   Between $1/u$ and $\log \log R$; $r = +.922$

The straight lines through these scatters were drawn freehand
rather than by any least squares formula. Equations expressing the
relation of $1/u$ to $f(R)$ were then calculated from the freehand lines.
In this case we believe such a procedure is preferable to the usual least
squares procedure. We do not wish to minimize the sum of the squares
of deviations of either variable alone from the values given by the
equation. Moreover, a freehand line through the data appears to us as
satisfactory as any form of "mean" or "orthogonal" regression.

The best expression for the marginal utility of real non-food ex-
penditures as a function of the amount of these expenditures appears
to be:

(11)

$$\text{a. } 1917 \text{ to } 1921 \quad U = \frac{0.10}{\log \log R - \log \log 154.77}$$

$$\text{b. } 1922 \text{ to } 1932 \quad U = \frac{0.74074}{\log R - \log 144.67}.$$

These are the equations plotted in Figure 5 together with a freehand
curve, which is very similar in both cases. In neither case, however,
should we consider either the mathematical or the freehand curves as
being exact but only as approximations based on limited ranges of
data. Extrapolation beyond the range of the data would have little
value.

From equations (11) the flexibilities of marginal utility with respect to amounts of real expenditure are:

a. 1917 to 1921 $\quad F = \dfrac{-\,.1886}{\log R(\log \log R - \log \log 154.77)}$

(12)

b. 1922 to 1932 $\quad F = \dfrac{-\,0.4343}{\log R - \log 144.67}.$

These equations give the following flexibilities:

Real expenditures for other than food (1913 dollars per capita)	Flexibility of marginal utility	
	1917–1921	1922–1932
150	−1.452	−0.772
200	−0.977	−0.623
250	−0.772	−0.547
300	−0.657	−0.496

These flexibilities check fairly well with those obtained by Frisch for the United States based on the 1918–19 budget study[13] if we bear in mind that per capita incomes covered in Frisch's study were higher than those used here representing an average for all groups of the population. Frisch estimated that the elasticity of the marginal utility of money varies from −0.6 to −0.3.[14] This result might be considered as a measure of the elasticity of the marginal money utility on the part of wage earners and low-to-medium salary earners and is somewhat lower than we get here for the population as a whole. This seems entirely reasonable. Money wages in the city have been consistently higher than the cash income of farmers. Also farmers as a group are usually known as careful spenders who "know the value of a dollar" and do not part with one as readily as do city workers.

The shape of the curve of marginal utility of real income is very similar to the shape of the curves found by Frisch both in the United States and in Paris. All available statistical evidence indicates that the elasticity of the marginal utility of real income is higher for small incomes and lower for high incomes. The function suggested by Ber-

[13] Frisch, *New Methods of Measuring Marginal Utility, op. cit.*

[14] The results recently attained by Frisch and reported in Publikasjon nr. 5 of the Universitets Økonomiske Institutt, Oslo, Norway, indicate an average flexibility of about −0.96 from 1919 to 1931. These results were obtained by a study of consumption and prices of meat and butter in the United States and are more directly comparable with the results of the present study than are his results based on the 1919 budget study.

nouilli is not satisfactory, since it gives for all incomes elasticities greater than unity, whereas it is apparent both from Frisch's study and from the present analysis that in the United States the elasticity is commonly less than unity. On the other hand, formulas (6) and (7) or modifications of these formulas appear to fit the observed data well.

Further facts needed.—If studies of the marginal utility of money are to be of the greatest value both from the standpoint of practical application and from the standpoint of theory it will be necessary to study wider ranges of income variations than have yet been covered. Tinbergen[15] remarks that it would be difficult, and somewhat cruel, to decide by experiment what is the shape of the curve of marginal utility of money beyond the usual range of incomes. This, luckily, is not necessary. There is plenty of variation in individual incomes. Many families are undoubtedly now living very close to the minimum of existence while others have incomes so great that an additional dollar can give them no great utility.[16] Further studies of the variation in money utility among groups of families classified by income may be very enlightening on this point, particularly if based on accurate data covering expenditures of a fairly large number of representative families over a period of years. Records of the expenditures of special groups such as ex-millionnaires who lost their money in the recent depression would also be extremely valuable in this connection, particularly if we could continue to get similar records while they get their next millions.

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[15] J. Tinbergen. "Annual Survey of Significant Developments in General Economic Theory," ECONOMETRICA, January, 1934.

[16] In studying groups with very high incomes it may be necessary to modify our concepts of utility and perhaps to adopt something like Frisch's *A*, *B*, and *C* concepts in *New Methods, op. cit.*

LA DYNAMIQUE DE LA CIRCULATION

Par Luigi Amoroso

AVANT-PROPOS

Le rapport que j'ai l'honneur de présenter au IV Congrès de la Société Econométrique est la synthèse d'études et de recherches que j'ai poursuivies sur les problèmes de la dynamique monétaire. Quelques-uns des résultats qui paraissent ici ont été déjà publiés ailleurs, et précisément: (a) dans les Chapitres II et III de mon *Contributo alla teoria matematica della dinamica economica,* publié dans le deuxiéme volume de la *Nuova Collana di Economisti* éditée par les soins de MM. G. Bottai et C. Arena (Torino, U.T.E.T., 1932); (b) dans les Chapitres VIII et XII de ma *Dinamica dei prezzi,* résumé d'un cours professé à l'Université de Rome pendant l'année 1932-33 (édition du "Gruppo Universitario Fascista dell'Urbe"). Toute conclusion de ces études qui ne s'accorderait pas avec les résultats que je vais exposer doit être considérée comme dépassée.

Dans l'introduction à mon *Contributo* (*op. cit.,* p. 421) j'écrivais: "Je rassemble dans ce mémoire quelques idées sur la théorie mathématique de la dynamique économique que j'ai présentées dernièrement dans d'autres articles ayant paru sporadiquement, sous des formes et dans des lieux différents. J'ai coordonné ces idées en un complexe plus organique, en ajoutant quelques développements nouveaux; c'est seulement en cédant aux insistances courtoises de MM. les Directeurs de ce Recueil que j'ose publier ici ces résultats; je crois pourtant de mon devoir de déclarer ouvertement que je considère mes recherches comme étant encore en cours d'élaboration; par conséquent je ne saurais reconnaître à mes résultats actuels un caractère de systématisation théorique, quand même on donnerait à cette expression le sens restraint dans lequel seulement elle peut être employée dans l'histoire des doctrines économiques. Les résultats que je présente aujourd'jui ne sont qu'une étape, une position d'attente, une tranchée utile pour coordonner ses idées et reprendre haleine en vue d'un nouveau progrès."

Si la passion dans laquelle j'ai poursuivi mes recherches ne m'aveugle point, je pense avoir accompli ce progrès, pour ce qui a trait à la dynamique de la circulation; et j'en fais l'objet du présent rapport. Celui-ci s'inspire, dans ses concepts fondamentaux, non seulement des deux travaux indiqués ci-dessus (et de ceux dont ils dérivent), mais encore de la Relation que M. A. De Stefani et moi-même nous avons présentée à la Conférence Economique réunie à Londres en Mai 1933 par

l'initiative de l'Institut pour la Coopération Intellectuelle. Cette Relation, dont le titre est "La logique du système corporatif" est publiée dans les Comptes-Rendus de la Conférence (Padova: Cedam, 1934), et en outre dans la *Rivista Internazionale per le Scienze Sociali* (XLI année, IV fasc., Juillet 1933), dans le *Archivio di Studi Corporativi della R. Università di Pisa* (IV année, II fasc.) et dans les *Annali di Economia* de l'Université Bocconi de Milan (IX année). Tout ce qu'on peut y lire s'harmonise parfaitement avec la présente étude qui doit être considérée comme un développement de l'autre, et à laquelle on pourrait donner le titre de "Discipline Corporative de la circulation monétaire."

Quoique sur le point central de la politique monétaire mes idées soient opposées de celles de M. I. Fisher, le présent Rapport s'inspire aussi de quelques conceptions théoriques de cet éminent auteur, particulièrement de celles qui sont exposées dans le mémoire "Our Unstable Dollar and the So-called Business Cycle" (*Journal of The American Statistical Association*, June 1925), dans la *Theory of Interest* (New York: Macmillan, 1925), et dans le volume *Inflation* (London: G. Allen, 1933). Il s'inspire enfin aussi—quoique dans un moindre degré—des concepts que M. Keynes a exposés dans son *Treatise of Money* (London: Macmillan, 1930) et de l'idée animatrice de la construction du baromètre de la Harvard Economic Society.

I. RÉACTIONS PROPRES ET RÉACTIONS INDUITES

Nous concevons la dynamique de la circulation monétaire comme le résultat des actions et réactions réciproques des systèmes de l'industrie, du commerce, et de la banque.

En nous rapportant au schéma théorique que nous avons présenté dans notre "Logique du système corporatif," nous classons ici les réactions élémentaires de ces trois systèmes en trois groupes fondamentaux, soit: les *réactions d'inertie*, les *réactions mécaniques*, et les *réactions directrices*.

Aux réactions d'inertie nous pouvons donner aussi le nom de *réactions propres* en indiquant par là qu'elles ont lieu dans le secteur même de l'action; les réactions mécaniques et les réactions directrices pourront être appelées, par opposition, *réactions induites*.

Les réactions d'inertie se produisent constamment dans tous les secteurs du monde économique. Elles sont un reflet de la situation générale plutôt que des situations particulières des différents secteurs. Les conditions préexistantes, les intérêts constitués, l'effort qui est toujours nécessaire pour imprimer à une activité un sens ou un rythme différent, les pertes que toute démobilisation et tout changement entraînent, sont les liens par lesquels les forces d'inertie s'enracinent dans le terrain de l'activité économique. Elles représentent le poids

du passé qui impose des conditions à l'avenir; elles représentent aussi le moyen par lequel le processus historique domine la réalité vivante.

En général, toutefois, l'esprit d'innovation est assez fort pour retarder le mouvement en cours, que les forces d'inertie tendent à prolonger.

Les réactions induites se présentent par contre avec des caractères différents dans les différents secteurs: elles sont mécaniques dans le système de l'industrie, et directrices dans le système du commerce, tandis que dans l'action du système de banque les tendances mécaniques et directrices l'équilibrent mutuellement.

II. LES RÉACTIONS INDUITES DANS LE SYSTÈME DE L'INDUSTRIE

Les deux tendances fondamentales de l'économie de l'entreprise—la tendance à niveler entre eux le coût et le prix, laquelle guide la direction commerciale de l'entreprise dans le choix des quantités qu'il faut produire, et la tendance à réduire les coûts à un niveau minimum, laquelle guide la direction technique de l'entreprise dans la combinaison des facteurs de la production—se manifestent ici à travers la réaction de l'industrie à l'action du commerce et à l'action de la banque respectivement.

La première des tendances susdites a son origine dans le principe de la propriété privée en conséquence duquel la production, qui se fait aux risques et aux périls des individus, est encouragée par l'espoir des profits et retenue par la crainte des pertes. Aux périodes de hausse des prix, la valeur du produit augmente plus rapidement que le coût de revient; par conséquent la conditions des plus hauts gains espérés est réalisée et l'industrie s'épand, élargit ses établissements, se grée pour produire davantage. Et pour des raisons opposées, elle démobilise aux périodes de baisse des prix.

Ce sont partant les prix *croissants* (et non pas les prix les plus élevés) qui déterminent la tendance à un niveau élevé de la production future; et ce sont les prix *décroissants* (et non pas les prix les plus bas) qui déterminent la tendance opposée.

C'est justement parce qu' elle donne lieu non pas à des prix croissants, mais seulement à des prix élevés (par rapport à la situation précédente) que la dévaluation monétaire ne peut pas inciter la production *d'une façon durable*. Pour que l'incitement ne fût pas éphémère, il faudrait que la première dévaluation fût suivie d'une deuxième, la deuxième d'une troisième, et ainsi de suite. Mais tout cela est évidemment absurde; et d'ailleurs c'est justement dans les suites de dévaluations successives qu'il faut rechercher l'origine de la tragédie des inflations des périodes de guerre et de l'après-guerre.

La réaction de l'industrie à l'action de la banque se produit dans un

sens opposé de celui de l'action; en effet, l'augmentation des coûts provoquée directement par la hausse du taux d'intérêt ne trouve pas de compensation dans l'augmentation de la valeur du produit, puisque le mouvement du taux d'intérêt—considéré séparément, comme il doit l'être ici—n'a pas d'influence sur cette valeur. Un mouvement de hausse du taux d'intérêt détermine directement une tendance au ralentissement de la production *future;* et un mouvement de baisse du taux d'intérêt détermine une tendance opposée.

III. LES RÉACTIONS INDUITES DU SYSTÈME DU COMMERCE

Ces réactions expriment les deux tendances fondamentales du marché: la tendance à niveler entre eux le prix du produit et le volume de la production future, et la tendance à niveler entre eux le prix des capitaux (cours des titres) et le flux des revenus futurs.

L'une de ces tendances représente la réaction du commerce à l'action de l'industrie, l'autre la réaction du commerce à l'action de la banque.

C'est la première qui a sans doute le plus de signification; c'est au moyen de cette réaction que le marché escompte *immédiatement,* dans les prix, les variations futures de la production, *telles qu'elles sont prévues pro tempore;* elle exprime la loi de l'offre et de la demande, conçue dans un sens dynamique ou différentiel, c'est-à-dire comme équation différentielle liant les prix, les quantités et leurs dérivées. Le marché tend précisément à établir un niveau de prix haut ou bas selon qu'il prévoit un ralentissement ou une accélération de la production future. De cette façon le marché tend à imprimer au système commercial une réaction dans un sens contraire de celui de l'action prévue du système industriel.

La réaction du commerce à l'action de la banque s'exprime théoriquement dans le principe de la capitalisation des revenus en conséquence duquel la valeur des capitaux (prix des terres et des bâtiments, cours des actions et des obligations) se nivelle sur la valeur des revenus correspondants—*tels que ceux-ci sont prévus pro tempore*—capitalisés au taux d'intérêt du marché—*tel qu'il est prévu pour le futur le plus proche*—et puisque la valeur capitalisée d'une rente temporaire ou perpétuelle varie dans un sens contraire de celui du taux d'intérêt sur la base duquel le calcul de la valeur capitale a été effectué, il s'en suit que la réaction qui est imprimée de cette façon au système commercial a un sens contraire de celui de l'action prévue du système de la banque.

Les deux réactions que nous venons de considérer ne se produisent pas d'une manière mécanique, par l'action autonome des forces autorégulatrices du système économique; *c'est dans la faillite des processus de nivellement correspondants que réside la contradiction la plus éclatante*

du système libéral. Les concentrations industrielles, les sociétés en chaîne, les gains produits par la conjoncture économique, les manœuvres de la Bourse, les fluctuations des cotes des titres, les divergences observées entre les cours des actions et les conditions réelles de l'industrie, sont les signes les plus évidents de cette contradiction. C'est justement parce que la surveillance d'une volonté centrale est nécessaire pour empêcher les déformations du marché, l'asservissement de la politique à l'économie, l'empire de la plutocratie, que les réactions du système commercial, telles que nous les avons considérées ici, méritent d'être qualifiées de *directrices*.

IV. LES RÉACTIONS INDUITES DANS LE SYSTÈME DE LA BANQUE

Les réactions induites du système de la banque expriment le nivellement entre le taux d'intérêt et les prix dans un passé tout récent et, respectivement, le nivellement entre le taux d'intérêt et la production dans un futur immédiatement proche. Ces deux actions sont convergentes dans le sens que c'est au moyen d'elles que la banque modère l'élan du marché quand celui-ci montre une trop forte inclination vers la hausse, ou qu'elle l'excite au contraire quand le marché montre une trop forte inclination vers la baisse.

La première des tendances susdites représente la réaction du système de banque à l'action du système commercial; elle est l'expression du principe fishérien d'après lequel le taux d'intérêt semble être élevé alors qu'en réalité il est bas, et semble être bas alors qu'en réalité il est élevé. C'est une chose bien connue que la raison de cette inversion réside dans le fait qu'aux périodes de hausse des prix—soit de dévaluation de la monnaie—la somme qui est versée pour solder une dette a un pouvoir d'achat moindre que celui de la somme qui fut versée au moment où le prêt fut accordé; le prêteur subit par là une perte, tandis que l'emprunteur obtient un gain, et cette double compensation induit les deux parties contractantes à s'accorder sur un taux d'intérêt élevé. La situation est renversée pendant les périodes de baisse des prix, alors que la somme remboursée par l'emprunteur a un plus fort pouvoir d'achat que celle qui fut versée par le prêteur, et cela induit les parties contractantes à s'accorder sur un bas taux d'intérêt.

La hausse des prix provoque donc mécaniquement la tendance vers de hauts taux d'intérêt; la baisse des prix provoque une tendance vers de bas taux d'intérêt; et cela signifie que la réaction de la banque à l'action du commerce se produit dans le sens même de cette action.

La réaction de la banque à l'action de l'industrie est une réaction directrice et s'exerce elle aussi dans le sens même de l'action. Elle exprime la loi de l'offre et de la demande, puisque l'accroissement de la production implique l'élargissement des installations et est escomptée

par la banque qui tend, en correspondance, à élever le coût de l'argent. La banque tend par contre à maintenir ce coût dans des limites modestes lorsqu'on peut prévoir un ralentissement de la production, c'est-à-dire une démobilisation des installations. Il y a là l'un des aspects fondamentaux de ce que l'on appelle la *manœuvre du taux d'intérêt;* dans des conditions normales, celle-ci est justement l'instrument par lequel le système de la banque dirige la production et le mouvement économique en général.

V. LES DIFFÉRENCES DE PHASE

Les réactions d'inertie sont l'expression de forces agissant d'une manière continue, et partant elles se produisent en même temps que les actions qui les déterminent; les réaction directrices se produisent avec une avance de phase, les réactions mécaniques avec un retard de phase.

L'avance ou le retard de phase sont minima—un mois environ—dans le cas des actions et réactions réciproques des systèmes du commerce et de la banque; et cela parce que, d'un côté, le mouvement du taux d'intérêt n'est pas susceptible d'être prévu à une plus longue échéance, et que, de l'autre côté, il réflète immédiatement les interférences du mouvement des prix.

L'avance ou le retard de phase sont maxima—cinq ou six mois environ—dans le cas des actions et réactions réciproques des systèmes industriel et commercial, parce que l'adaptation de la production à un mouvement déterminé des prix suit ce mouvement et se réalise pendant une période ayant à peu près la même durée que le cycle de la production; et une période à peu près égale est nécessaire pour que les prix actuels se nivellent à la production future.

L'avance ou le retard de phase ont enfin une valeur moyenne— quatre ou cinq mois environ—dans le cas des actions et réactions réciproques des systèmes de l'industrie et de la banque: d'un côté il n'est pas nécessaire que les capitaux nouveaux qui sont demandés à la banque pour les installations nouvelles soient tous liquides au moment où l'agrandissement des établissements est décidé, et ils peuvent être versés graduellement; d'autre côté, l'augmentation des coûts de production qui a lieu en conséquence de la hausse du taux d'intérêt, se produit elle aussi d'une façon graduelle; et cela parce que les termes d'intérêt et d'amortissement relatifs aux capitaux dernièrement employés font sentir d'une façon graduelle leur poids sur le coût du produit.

Si l'on indique en général par

$$\omega_1, \ \omega_2, \ \omega_3,$$

les trois intervalles minimum, moyen et maximum, on peut écrire en tout cas l'égalité:

$$(1) \qquad\qquad \omega_1 + \omega_2 = \omega_3.$$

VI. REPRÉSENTATION MATHÉMATIQUE TOTALITAIRE

Les mouvements des trois systèmes que nous venons de considérer se déterminent donc réciproquement dans un ensemble d'actions et de réactions dont l'analyse mathématique permet de donner une synthèse totalitaire.

Dans ce but nous prenons comme expression de la situation industrielle l'indice T de la production, compréhensif de la production industrielle proprement dite et de la production agricole; nous prenons comme expression de la situation commerciale l'indice P des prix, compréhensif du prix des marchandises et du prix des capitaux (cours des actions et des obligations); nous prenons enfin comme indice de la situation de la banque le taux d'intérêt j, exprimant en même temps le coût et le prix de l'argent, c'est-à-dire le taux d'escompte et le taux d'intérêt proprement dit.

Si t indique le temps, T, P, j, seront des fonctions de t et leurs valeurs empiriques dans la période d'observations seront données par trois séries de nombres indices.

Mais ce n'est point la même chose que de dire que les prix sont hauts ou bas, ou qu'ils vont augmenter ou diminuant. A côté de l'indice P, qui représente le niveau général des prix, il faut considérer un deuxième indice qui représente la *vitesse* du mouvement des prix et qui n'est que la dérivée de la fonction P par rapport au temps; soit, en adoptant les symboles usuels, P'. Nous devons de même considérer la *vitesse* du mouvement du taux d'intérêt, soit la dérivée de la fonction j par rapport au temps.

L'indice T de la production est un flux et partant il represente lui-même une vitesse. Son mouvement sera donc exprimé par une *accélération* et sera représenté d'une façon analogue par la dérivée T' de T par rapport au temps.

Les mouvements des trois indices dérivés,

$$T', \qquad j', \qquad P',$$

représentent respectivement les actions du système de l'industrie, du commerce et de la banque.

Les mouvements que dans les différentes combinaisons possibles ils induisent directement dans les trois indices primitifs,

$$T, \qquad j, \qquad P,$$

expriment les réactions correspondantes.

Ayant établi ces concepts, nous allons maintenant examiner d'une façon particulière les réactions du système commercial, soit:

la réaction propre d'inertie,

la réaction induite du système de banque,

la réaction induite du système industriel.

La première de ces réactions se produit en même temps que l'action et dans un sens opposé du sens de celle-ci; la deuxième se produit dans un sens opposé de celui de l'action qui la détermine et avec un retard de phase égal à ω_1; la troisième se produit aussi dans un sens opposé de celui de l'action et avec un retard de phase égal à ω_3. Cela signifie qu'il y a:

une corrélation inverse entre $P(t)$ et $P'(t)$,

une corrélation inverse entre $P(t)$ et $j'(t+\omega_1)$,

une corrélation inverse entre $P(t)$ et $T'(t+\omega_3)$.

Cette dépendance triple est exprimée d'une manière synthétique par l'équation:

$$(A) \qquad P(t) = m_1 - a_{11}P'(t) - a_{12}j'(t + \omega_1) - a_{13}T'(t + \omega_3),$$

a_{11}, a_{12}, a_{13}, m_1, étant des constantes positives.

L'équation (A) représente d'une manière totalitaire les réactions du système commercial; nous l'appellerons donc *équation de la Bourse* (ou du commerce).

On peut écrire d'une façon analogue l'équation de la banque:

$$(B) \qquad j(t) = m_2 + a_{21}P'(t - \omega_1) - a_{22}j'(t) + a_{23}T'(t + \omega_2),$$

qui représente toutes les réactions du système de la banque; et *l'équation de l'usine* (ou de l'industrie):

$$(C) \qquad T(t) = m_3 + a_{31}P'(t - \omega_3) - a_{32}j'(t - \omega_2) - a_{33}T'(t),$$

qui représente toutes les réactions du système industriel.

Dans ces équations les coefficients a_{rs} pour lesquels on a $r \neq s$ sont tous des constantes positives qui y paraîssent précédés par le signe d'addition ou de soustraction selon que la corrélation élémentaire (induite) à laquelle chaque coefficient se rapporte est directe ou inverse.

Les coefficients a_{11}, a_{22}, a_{33}, des corrélations propres doivent être considérés comme étant égaux entre eux parce que les mouvements d'inertie auxquels ils se rapportent sont des reflets de la situation générale plutôt que de situations particulières des différents secteurs. Si l'on indique par a leur valeur commune, on a:

$$(2) \qquad\qquad a_{11} = a_{22} = a_{33} = a$$

a étant une constante positive.

Enfin les coefficients m_1, m_2, m_3, sont des constantes non essentielles, dépendant du choix des bases des indices considérés.[1]

Si l'on pose:

$$(3) \quad x = P(t) - m_1; \quad y = j(t + \omega_1) - m_2; \quad z = T(t + \omega_2) - m_3;$$

si l'on se souvient que, en conséquence de (1), il est:

$$\omega_1 + \omega_2 = \omega_3,$$

et si l'on tient compte des égalités (2), on peut écrire les trois équations ci-dessus sous la forme:

$$(D) \quad \begin{aligned} x &= -ax' - a_{12}y' - a_{13}z', \\ y &= a_{21}x' - ay' + a_{23}z', \\ z &= a_{31}x' - a_{32}y' - az', \end{aligned}$$

dans laquelle elles constituent un système de trois équations différentielles homogènes à coefficients constants; *un système qui représente la dynamique de la circulation dans un sens totalitaire* et dans les limites d'approximation dans lesquels cette dynamique est exprimée par le mouvement des indices que nous avons considérés.

VII. EXPLICATION DU MOUVEMENT CYCLIQUE

Au moyen d'une intégration du système (D) on obtient le mouvement théorique des trois indices T, j, P. Afin d'effectuer cette intégration il faut poser, comme on sait:

$$(4) \quad x = Ae^{t/\mu}; \quad y = Be^{t/\mu}; \quad z = Ce^{t/\mu};$$

où e est la base des logarithmes naturels, A, B, C, μ, sont des constantes qui doivent être déterminées de manière à ce que les équations du système (D) soient vérifiées. En effectuant la substitution l'on parvient à un système d'équations linéaires en A, B, C, qui sont compatibles sous la conditions que μ satisfasse à la suivante équation du troisième degré:

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[1] On peut supposer, d'une façon plus générale, que m_1, m_2, m_3, soient des fonctions linéaires de la masse monétaire M (soit de la quantité de la monnaie légale et de la monnaie de banque). Et dans cette hypothèse les équations (D) représentent aussi les réactions qui ont lieu dans les systèmes de l'industrie, du commerce et de la banque en conséquence des variations de la quantité de la monnaie; elles en viennent partant à exprimer la synthèse de la théorie quantitative de la monnaie avec la théorie du mouvement cyclique.

Une généralisation de ce genre ne donne lieu à aucune difficulté mathématique pour l'intégration des équation (D), puisque la quantité M doit être considérée comme une fonction du temps.

$$\begin{vmatrix} -a - \mu & -a_{12} & -a_{13} \\ a_{21} & -a - \mu & a_{23} \\ a_{31} & -a_{32} & -a - \mu \end{vmatrix} = 0$$

En posant

(5) $$\lambda = -(a + \mu)$$

on peut écrire l'équation ci-dessus sous la forme

(6) $$\lambda^3 + p\lambda + q = 0$$

où

$$p = a_{12}a_{21} + a_{23}a_{32} + a_{13}a_{31},$$
$$q = a_{13}a_{21}a_{32} - a_{31}a_{12}a_{23}.$$

Puisque $p > 0$, l'équation (6) n'admet qu'une seule racine réelle, et par conséquent les intégrales x, y, z, du système (D) ont des expressions de la forme suivante:

$$Ae^{rt} + Be^{st} \cos\left[\frac{2\pi t}{\sigma} + q\right].$$

Le mouvement des trois indices fondamentaux se présente donc comme résultant de la composition de *mouvements évolutifs* et de *mouvements cycliques*.

Les constantes A, r, représentent respectivement l'ampleur et l'intensité du mouvement évolutif.

Les constantes B, σ, q, représentent respectivement l'ampleur, la période, la phase, du mouvement cyclique.

La constante s représente le coefficient d'amortissement des oscillations cycliques.

Les quantités r, s, σ, sont invariantes pour les trois indices.

Les phases du mouvement dépendant des constantes ω_1, ω_2, dont nous avons indiqué plus haut la signification, et des valeurs des coefficients paraissant dans les équations fondamentales de la bourse, de la banque et de l'usine, sont par contre différentes pour les trois indices.

La manière dont nous avons déduit ici le mouvement cyclique montre *que aucune manœuvre monétaire ne pourrait le renverser*. Nous avons déjà éclairé au paragraphe 2 de ce Rapport, la contradiction qui est au fond même da la thèse qu'une dévaluation monétaire puisse engendrer une expansion *durable* de la production. Nous avons ici une nouvelle démonstration *à posteriori* de cette contradiction, puisque le caractère oscillatoire du mouvement apparait comme étant *le résultat du contraste des actions et réactions fondamentales* de l'industrie, du commerce

et de la banque. Ces actions et réactions ont leur racine dans le régime juridique de la propriété privée et dans la liberté des négotiations. C'est une conséquence de la première que la production actuelle se nivelle aux prix et aux coûts de revient, au risque et au profit des individus; et c'est une conséquence de la deuxième que les prix se proportionnent à leur tour à la production future et au rendement actuel des capitaux, toujours au risque et au profit des individus. L'action de la banque peut s'exercer seulement dans un sens marginal, en rendant plus aisé ou en retardant, selon les cas, ces deux nivellements en vue d'amorcer les fluctuations trop violentes.

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DEMAND IN RELATION TO THE INCOME CURVE[1]

By Jan Wisniewski

1. General notions.—Assume that we have to do with a homogeneous
population, whose distribution with respect to the individual (or per
family) income x is described by the function $y(x)$. For a given price
of a commodity A, the total expenditure of a person for this commodity
is a function of his income $w_A(x)$, making the total expenditure of the
population $\int_{x_0}^{\infty} w_A(x)y(x)dx$, x_0 being the minimum income. It should
be noted that (1) w is really a regression function, although for theoret-
ical purposes we shall regard it as a function proper; (2) the commodity
A may be a one-price commodity, such as crystalline sugar, or a many-
priced one, such as cigarettes; in the latter case, when speaking of a
"given price," we really think of a given price system.

The function w, which we may call—according to Frisch's proposal
—an Engel function, is, of course, limited in value by the inequality
$0 \leqq w_A(x) < x$. As x increases, $w_A(x)$ may increase in proportion to x,
more slowly than x, or faster than x. Accordingly, we shall say that the
elasticity of expenditure for A with respect to income is equal to, less,
or greater than, unity, the elasticity being defined as follows:

$$e = \frac{dw}{dx} \cdot \frac{x}{w}.$$

The above concepts are well known in economic literature (Frisch,
Marschak).

2. Distribution of consumers' expenditures.—The function y gives the
distribution of consumers according to their incomes. We may also
investigate the distribution of their income-sums according to the
height of incomes. This distribution, used in the construction of Lorenz
curves, is

$$(2) \qquad\qquad u(x) = xy(x).$$

As we know, the Lorenz curve is the graph of an implicit function

$$(3) \qquad \frac{\int_{x_0}^{x_i} u(x)dx}{\int_{x_0}^{\infty} u(x)dx} = F\left(\int_{x_0}^{x_i} y(x)dx \right).$$

Let us consider the distribution of consumers' dollars expended for a

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[1] The writer is indebted to Mr. M. Kalecki for a critical reading of the manu-
script of the present paper.

given commodity A. Here we have the function

$$(4) \qquad\qquad z(x) = w_A(x)y(x).$$

For the special case when $e = 1$, the distributions (2) and (4) are of the same shape.

As we have mentioned, the function (2) is used (cumulatively) for constructing the curve of concentration of incomes (Lorenz curve). In an analogous way, we can construct a curve of concentration of distribution of expenditures for A. This curve is the graph of the implicit function

$$(5) \qquad\qquad \frac{\int_{x_0}^{x_i} z(x)dx}{\int_{x_0}^{\infty} z(x)dx} = G\left[\int_{x_0}^{x_i} y(x)dx\right].$$

The concentration of expenditures is greater than, equal to, or less than, the concentration of incomes, according as $e \gtreqless 1$ (provided $dw/dx \geq 0$).

The curve z may be described by characteristics just like any frequency function. The arithmetic average $\int_{x_0}^{\infty} xz(x)dx / \int_{x}^{\infty} z(x)dx$ gives the average value of incomes when they are weighted not only by their number but by the expenditure for A of their recipients also. The mode, again, defines the value of x from which comes the most intensive demand for A, or, in other words, the typical consumer (typical as to his income). This concept might be of practical interest for "many-priced" commodities. Suppose the demand varies qualitatively as income rises, "quality" being identified with the price of a given variety of A. The price at which a consumer of income x buys A is $p(x)$. This is, of course, again a regression function, but we shall regard it as a function proper.

The frequency function of consumers' dollars expended for A with respect to the price paid for A is

$$(6) \qquad\qquad v(p) = z(x)\frac{dx}{dp}.$$

The mode of this function gives the typical price of A, or, in other words, that variety of A for which the consumers' expenditure is the greatest (assuming p to be a continuous function of x). The mode is found by solving

$$(7) \qquad\qquad \frac{dz}{dx}\left(\frac{dx}{dp}\right)^2 + z(x)\frac{d^2x}{dp^2} = 0$$

for p. Of course, we may also find the value of x corresponding to p defined as above.

It is worth noting that, in the system $y(x)$, $z(x)$, $w(x)$, $p(x)$, $v(p)$, three functions only are mutually independent. Therefore, knowing any three of them[2], we are able to find the remaining two. If we know (empirically) more than three, we have a means of checking our estimates.

3. The demand for luxuries.—We define as luxuries those goods that are bought only by consumers from the upper part of the income curve, i.e., whose incomes $x \geqq a\, x_0$ (for a given commodity A), x_0 being the lower limit of incomes.

It is currently assumed that the demand for luxuries is very elastic. Ignoring the problem of demand elasticity proper (with respect to prices), we shall attempt an analysis of the behavior of expenditures for luxuries when the income curve is subject to a deformation, and especially when all incomes suffer a marked decrease, as in a time of business depression. We shall refrain from the assumption that incomes high up in the income curve undergo a sharper decline than the average, notwithstanding the great likelihood of such a shape of things.

Suppose that all incomes shrink in the same ratio, $1:mn$, where m is the ratio of real income in the new period to that of the base period (which might be identified with an index of production volume, including services), and n is the price index. For the sake of simplicity, we shall further conduct our analysis in terms of real (deflated) prices, so that the ratio of the reduction of incomes becomes $1:m$. The shape of the income distribution is now such that

$$(8) \qquad\qquad y(x') = y(x),$$

where $x' = mx$. It should be noted that the left-hand side of (8) gives the frequency from x' to $x' + dx'$, the right-hand side from x to $x + dx$. Therefore, we may write the new income distribution function with respect to x,

$$(9) \qquad\qquad y'(x) = \frac{1}{m} \cdot y\left(\frac{x}{m}\right),$$

where both y and y' represent frequencies from x to $x + dx$.

Assume that money prices of A fall in the same ratio as the general index, its real prices remaining unchanged. The function $w_A(x)$ remains unchanged, too.

Let us investigate what will be the consequences of the above described deformation of the income curve. In so far as demand for a given ("luxurious") commodity is concerned, three separate magnitudes might be discussed. The first of them is the number of those able to purchase A. There is no doubt as to the shrinkage of this number (taking the total population number as constant). Indeed,

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[2] Except, however, the triads $y(x)$, $w(x)$, $z(x)$, and $z(x)$, $p(x)$, $v(p)$.

$$(10) \qquad \int_a^\infty y'(x)dx = \int_{a/m}^\infty y(x)dx < \int_a^\infty (y)x\,dx.$$

The integrals (10) are of special interest when we have to do with such goods as might be assumed to be purchased by each consumer in a constant amount (while of varying quality), such as automobiles.

The second one is the total income of the consumers in question. This suffers a twofold reduction, because (1) the real value of the national income, as a whole, is now smaller in the ratio $1:m$, and (2) the share of those consumers is now smaller than in the base period. Nevertheless the share in the national income is reduced less than the relative number, as is evident from the Lorenz curve. The relation is as follows:

$$(11) \qquad \int_a^\infty xy'(x)dx = m\int_{a/m}^\infty xy(x)dx < \int_a^\infty xy(x)dx.$$

The question may be legitimately put whether

$$(12) \qquad \frac{m\int_{a/m}^\infty xy(x)dx}{\int_a^\infty xy(x)dx} \gtrless \frac{\int_{a/m}^\infty y(x)dx}{\int_a^\infty y(x)dx}.$$

This question cannot be answered generally, depending on the shape of the function $y(x)$ and on the position of the interval $a - a/m$. For a Paretian distribution ($y = cx^{-\alpha}$) the sign of equality holds true for any given a and m. We may call "sub-Paretian" such distributions (or definite intervals thereof) as satisfy the sign $<$ in (12), and "super-Paretian" such as give $>$.

The elasticity, e, for luxurious goods must necessarily be greater than one, at least in a certain interval. Therefore, we must introduce the function w and the integral

$$(13) \qquad \int_a^\infty w_A(x)y'(x)dx < \int_a^\infty w_A(x)y(x)dx.$$

Not knowing the form of w, we cannot directly estimate the effect of the reduction of incomes on the total expenditure for A. Nevertheless, as a first approximation, we may assume that w is of the form

$$(14) \qquad w_A(x) = q(x - a)$$

where $q < 1$. Here, again, the question rises whether

$$(15) \qquad \frac{\int_a^\infty w_A(x)y'(x)dx}{\int_a^\infty w_A(x)y(x)dx} \gtrless \frac{\int_a^\infty xy'(x)dx}{\int_a^\infty xy(x)dx}.$$

Assuming (14) to be true, it can easily be proved that the sign of (15) is the same as of (12).

As a further approximation, we may express w_A as

$$(16) \qquad w_A(x) = \sum_{i=1}^{k} b_i(x-a)^i,$$

where all b's are positive. Then we have

$$(17) \qquad \int_a^\infty w_A(x)y'(x)dx = \sum_{i=1}^{k} b_i m^i \int_{a/m}^\infty \left(x - \frac{a}{m}\right)^i y(x)dx.$$

In order to satisfy the condition $w_A(x) < x$, it may be necessary to change in (17) the upper integration limits from ∞ to mX and X, X being a practical upper limit of incomes.

In general, the greater e (as defined by 1), the greater the importance of the positively valued higher degrees of $x-a$ in (16), and the more likely the sign $<$ in the inequality (15).

Numerical example.—The Pareto curve does not give particularly good fits when applied to actual income distributions. The so-called logarithmic frequency curve,

$$(18) \qquad y = \frac{1}{\alpha(x-x_0)\sqrt{2\pi}} e^{-(1/2\alpha^2)lg^2[(x-x_0)/g]}, \; e = 2.718\cdots$$

proved to be much superior. Especially in so far as the "upper tails" of income distributions are concerned, the fits seem to be excellent (cf. the works of Gibrat, van der Wijk, and myself). Assuming the following values of the parameters: $x_0 = 0$, $\alpha = \sqrt{1/2}\, lg_e\, 10$ (this value is considered by Gibrat as a sort of normal), $g = 1$ (i.e., we measure the income in terms of the median), and the ratio of reduction of incomes $m = 3/4$, we get the following numerical results:

Percentage the purchasers of A are of total population (in the base period)	1%	0.1%	0.01%
a in terms of median income	44.1	153.1	427.0
Ratio of reduction:			
of the number of purchasers of A	0.548	0.468	0.410
of their total income	0.532	0.484	0.428
of their total expenditure for A / assuming $w_A(x) = q(x-a)$ /	0.518	0.504	0.460

From the above table, it is evident how violent is the reaction of expenditures for luxury goods in case of a general reduction of incomes. We may also note that the distribution in question is sub-Paretian in the first interval and super-Paretian in the two latter. The writer would like to stress the point that the data used in construction of the table, while fictitious, are none the less realistic.

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DIFFERENTIAL PRICE BEHAVIOR AS A SUBJECT
FOR COMMODITY PRICE ANALYSIS

By Holbrook Working

THERE IS A homely saying that the long way around may be the quickest way home. Many are the examples of the economy of indirection, whether in the affairs of everyday life or in the researches of science. It is especially in the more difficult of human endeavors that success is likely to turn on the use of a line of attack that is not the direct and obvious one.

Research into the laws governing commodity price movements may frequently advance more expeditiously, and often reach goals otherwise unattainable, by employing a line of indirect attack. One such line that is very commonly advantageous involves concentration of first attention on *differences* between properly chosen prices. These price differentials, often more conveniently described as *price margins* or *price spreads*, sometimes deserve to be treated in the form of percentage differences. In that case it may be more convenient to take as the subject of analysis *price ratios*.

Investigations of the behavior of price differences, or of price ratios, may be designated conveniently and meaningfully as studies of differential price behavior.

A logical analysis of the difficulties to be surmounted in quantitative investigations of commodity prices and of the means at hand yields a strong presumption that investigation of differential price behavior should provide a fruitful approach. From such a theoretical analysis of the methodological problem we shall proceed to a review of the fruits of application of the indicated general method in a variety of specific forms adapted to the requirements of diverse special cases. The most conclusive evidence on questions of methodology is provided by results in application.

A METHODOLOGICAL PROBLEM

Most of the quantitative research which has come to be known as "commodity price analysis" has been dominated by a body of ideas expressed in somewhat extreme form by that great pioneer in such work, H. L. Moore. In the concluding chapter of his *Forecasting the Yield and the Price of Cotton*, published in 1917, Moore wrote (pp. 172–173):

... economic theory has now reached the stage where, according to Professor Marshall, there is need of a "quantitative determination of the relative strength

of the different economic forces"; and, according to Professor Pareto, empirical laws must be derived from statistics for the double purpose of comparing them with known theoretical laws and of gaining bases for new theoretical developments.

The statistical theory of multiple correlation is perfectly adapted to these demands. No matter what may be the number of factors in the economic problem, it is specially fitted to make a "quantitative determination" of their relative strength; and no matter how complex the functional relations between the variables, it can derive "empirical laws" which, by successive approximations, will describe the real relations with increasing accuracy. . . .

Experience and more critical theoretical analysis have pretty well dispelled belief in the omnipotence of multiple correlation—a belief widely held for many years after Moore wrote so glowingly of its powers. The study of sampling errors has been disillusioning. We know now that the information which it is possible to extract through multiple correlation analysis of any individual price series is sharply limited as to quantity, and often highly questionable as to quality. Broadly, one may say that a multiple correlation analysis of such relatively short price series as are available for use can not measure the effect of more than three or four price influences, and that the measures afforded for some of these must usually be regarded as very rough and uncertain.

While we have been coming to a realization that, with the data available, the powers of multiple correlation analysis for revealing the laws of price are very limited, other developments have made the narrowness of that limitation appear increasingly severe. If the important factors bearing on the price of any one commodity were always very few in number and related to the price in very simple fashion, direct multiple correlation analysis might appear entirely adequate, even in the light of present general knowledge of its limitations. But intensive realistic study has revealed that, for some commodity prices, the number of factors that must be regarded as really important is rather large. Regressions are frequently curvilinear. The effects of price factors are often not independent, but joint. The factors, or at least the most suitable measures of them, are not known in advance, but remain to be determined; the character of the functional relationships between the factors, separately or jointly, and the price, is unknown and may not safely be assumed linear.

One development that has profoundly affected our appraisal of the problem of determining realistically the laws of price is the accumulation of evidence that the special problems of dynamic equilibrium are much more peculiar and more important than has generally been supposed. The common assumption appears to have been that to pass from the case of static equilibrium to that of dynamic equilibrium it is

necessary only to make allowance for what is commonly called "the effect of time," through some such process as the removal of trends. This assumption is not generally warranted. The difference between the static and the dynamic states, as regards laws of price behavior, involves much more than absence or presence of trends or other simple independent functions of time.

In the actual dynamic state, of course, there are trends to be considered, and these trends may commonly be regarded as reflecting effects that are independent of effects of other factors, in the sense that they may be removed by subtraction or by division without distorting the residual effects of the other factors. Occasionally, perhaps, cyclical tendencies are to be found properly subject to similar treatment. But, in the main, I believe, the cyclical tendencies which are prominent features of important commodity prices reflect the impact, under laws of dynamic equilibrium, of the same factors whose effects, under the better known laws of static equilibrium, have been matters of major concern in price analyses.[1] Merely to eliminate, by routine statistical treatment, these effects responsive to dynamic laws is to emasculate the price data. These effects, because they are so little understood, are even more worthy of study than the effects responsive to static laws. The general character of the static laws seems to be already fairly well known; in dealing with their manifestations, price analyses seem chiefly to contribute additional knowledge of the forms of the functional relationships and of their approximate relative importance. But, in dealing with the manifestations of dynamic laws, realistic price analyses have great opportunities for revealing important laws hitherto unrecognized.

Faced with the fact of exceeding difficulty and complexity in the problem presented for solution and of sharply limited powers in the instrument, it is logical to attempt to break down the main problem into components which may be individually capable of satisfactory treatment. If this proves feasible, one may then hope rather confidently to find means for integrating the results of the several related investigations into a comprehensive solution of the main problem. The investigation of differential price behavior in properly chosen cases appears highly promising from this standpoint.

RESULTS OF STUDIES OF DIFFERENTIAL PRICE BEHAVIOR

The foregoing considerations suggest a peculiarly significant light in which analyses of differential price behavior deserve to be viewed.

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[1] Illustrations may be found in the characteristics of price behavior discussed in "Cycles in Wheat Prices," *Wheat Studies of the Food Research Institute*, November 1931, especially pp. 18–44.

Have such analyses in fact proved advantageous for the discovery and elucidation of laws of price behavior?

It is at once apparent that the stated purpose in directing attention to differential price behavior calls for utilization of price series that are rather highly interdependent. The ideal situation is found where one series responds exactly like a second series to all the factors that bear on the second, and reflects, in addition, a fairly simple body of special influences. Even though response to common factors is not identical in the two series, however, price differences or price ratios may be highly favorable for study. All that is requisite is that the differential behavior be dominated by influences that in the separate series are obscured by more potent influences.

The idea of concentration on study of differential price behavior is exemplified in the very general practice of utilizing deflated prices for statistical analysis. The deflated price is a price ratio, and the immediate objective of its analysis is elucidation of differential price behavior. The fruitfulness of the approach through study of differential price behavior in this class of analysis is so well recognized that, if I were to discuss it at greater length, I should consider its weaknesses more in need of emphasis than its advantages.

What may prove a uniquely simple and direct application of the general plan of attack under consideration has been made in studies of hog prices. Analyses of supply-price relationships showed that the quantity of hog products moved into consumption appeared to be much more directly and simply related to retail pork prices than to wholesale hog prices. For this reason the determination of the influence of hog supplies upon wholesale prices of hogs can be much more precisely and accurately stated if it is separated into (1) an analysis of the relation of retail prices to quantities moved into consumption and to other disappearances of hog products, and (2) an analysis of the behavior of the margin between retail pork prices and wholesale hog prices.[1a]

For further illustrations of results of study of differential price behavior, I turn to results in the exceptionally difficult and complicated problems presented by wheat prices.

At the Food Research Institute, we have thus far brought to a stage of fairly satisfactory completion four major investigations of differential behavior of wheat prices. One deals with the relations between

[1a] These conclusions are based on work done in the Bureau of Agricultural Economics of the U. S. Department of Agriculture. The data appear in part in L. H. Bean and G. B. Thorne, "The Use of 'Trends in Residuals' in constructing Demand Curves," in *Journal of the American Statistical Association* March 1932, pp. 65–66.

prices of cash wheat in the United States and prices of Chicago wheat futures; two deal with interrelations among different Chicago wheat futures; and a fourth, chiefly the work of Dr. Robert D. Calkins, deals with interrelations among futures prices in three major wheat markets, Chicago, Winnepeg, and Liverpool.[2]

It is impossible within the limits of this paper to give a systematic presentation of the specific conclusions reached in these studies and their bearing on the general problem of explaining wheat price behavior. Resort must be had, instead, to a brief statement of major conclusions and their application in a few suggestive illustrations.

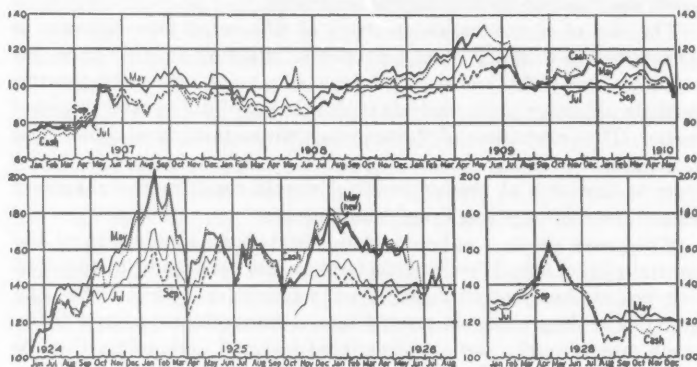Figure 1 presents graphically a record of wheat price behavior in

FIGURE 1.—Chicago Wheat Prices During Selected
Periods[a] (Cents per bushel)

[a] Closing prices on one day each week, usually Friday, compiled from Chicago *Daily Trade Bulletin*. Periods selected to illustrate price behavior under wide range of influences.

[2] Published in *Wheat Studies of the Food Research Institute* under titles and dates as follows: "The Post-Harvest Depression of Wheat Prices," November 1929; "Cycles in Wheat Prices," November 1931 (pp. 4–9); "Price Relations between July and September Wheat Futures at Chicago since 1885," March 1933; "Price Relations between May and New-Crop Wheat Futures at Chicago since 1885," February 1934; and "Price Leadership and Interaction among Major Wheat Futures Markets," November 1933.

Since the present paper was prepared we have published a further study, devoted largely to cash-future price relations, under the title, "Price of Cash Wheat and Futures at Chicago Since 1883," *Wheat Studies*, November 1934. It includes a set of charts which shows side by side in continuous chronological order weekly prices and spreads for the more important Chicago wheat price series from May 1883, and provides a useful supplement to the more specialized graphic presentations in the earlier monograph cited.

the Chicago market during periods that are particularly interesting for present purposes. The data presented are Friday closing prices of the lowest-priced contract wheat (cash) and of three wheat futures—the May, July, and September options. The simplest laws of differential price behavior represented here appear in the relations between the price of cash wheat and the price of the May future. Their differences in behavior reflect chiefly a *variable* seasonal tendency in the cash wheat price which is absent in the price of the future. Otherwise, the movements of the two prices are generally nearly identical. Broadly, the tendency is for the price of cash wheat to be depressed below the price of the future during late summer; to rise rather rapidly, relative to the future, during autumn and early winter; then to rise more slowly, until in May the two prices become identical, or nearly so.[3]

But, note the wide discount of cash wheat in the summer of 1907, the small discount in the summer of 1908, and the premium of cash wheat over the May future maintained from the summer of 1909 until the following April. The variable character of this cash-future relation is dependent primarily on the magnitude of domestic carryover of wheat and of the new domestic crop. It is significantly influenced also by other factors which have proved very difficult to reduce to statistical measurement. These other factors, which were specially important in 1909–10, are probably related to unusual disparities of judgment regarding the "reasonableness" of currently prevailing levels of wheat prices.

RELATIONS AMONG FUTURES

The prime clue to relations between prices of the May, July, and September futures is found in the fact that deliveries on May contracts must be made from wheat carried over from the previous summer, while wheat from a new harvest will be available for delivery on the two later futures—rather sparingly available in July and freely available in September. The May is called an "old-crop future," the July and September are both called "new-crop futures," although the behavior characteristics of the July are intermediate between those of the other two futures. The price of the May future ranges

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[3] For any serious study of price spreads or price ratios in graphic form, it is essential that they be plotted from a horizontal base rather than left, as in Figure 1, to be judged as margins between lines of varying slope. The form of presentation employed in Figure 1 is most appropriate for consideration of the relation between differential price behavior and absolute price behavior, with which this paper is chiefly concerned. In a presentation of the subject on a larger scale, this chart would be supplemented by two others, one designed specifically to show behavior of the spreads between prices of cash wheat and prices of the futures, the other designed specifically to show behavior of spreads between the different futures.

above the prices of new-crop futures if supplies of wheat for the expiring season are short, as in October–May of most of the years represented in Figure 1, and close to the prices of the new-crop futures, or below them if supplies are large, as in January–May 1907. The price spreads among these three futures, best expressed in percentage terms, are indeed rather closely related to the total supply of old wheat found to remain in the United States at the end of the season, commonly taken as July 1.

If supplies of wheat for the current season are regarded as excessive, some fraction of the supply must be considered as having value only for storage for final sale during the next season or later. The price of this marginal fraction and, therefore, the price of the whole supply, is dependent upon anticipated price at a relatively distant future date. In these circumstances, there is but one class of major price factor, namely, that which bears on fairly distant price expectations.

If supplies of wheat for the current season are regarded as notably short, three distinct classes of price influence may require consideration. The class or classes of influences in operation at any time may be determined very readily, as a rule, on the basis of peculiarities of differential price behavior associated with each class:

1. Influences bearing on prices anticipated for the following season affect prices of all futures under conditions of shortage as well as under conditions of surplus. The several futures tend to show equal percentage price response to such influences.

2. Influences related chiefly to appraisals of the degree of shortage of old-crop supplies tend to be accompanied by a particular set of differential price changes. Setting the consequent change in price of the May future at unity (100 per cent), the normal relationship may be represented by the following tabulation:

Change in price of May wheat.................   100 per cent
Change in price of July wheat....'..........   41 per cent
Change in price of September wheat...........   12 per cent

3. Influences chiefly related to the speculative developments associated with corners and squeezes, which have been common in years of short supplies, generally distort the relationships among prices of the several futures.

The record of wheat prices shown in Figure 1 exhibits several conspicuous price movements whose interpretation is facilitated by these generalizations. These may better be considered in the order in which the generalizations have been listed above, rather than in chronological order.

1. The sharp price rise of April–May 1907 clearly implied an abrupt change in price anticipations for the coming season. Old-crop supplies at this time were abundant and only anticipations for the coming

season could have been potent price factors. The same class of influence operating under broadly similar conditions are observed in the price movement of February–July 1928.

Obvious illustrations of dominance of this class of influences under conditions of *shortage* of old-crop supplies appear in the price movements of April–June 1925 and November 1925 to March 1926. In April–June 1925, crop developments in the United States were potent market influences; beginning in October 1925, Argentine crop developments assumed dominance. A less obvious, but correspondingly more significant indication of prominence of long-range price judgments as market factors is to be found in price movements during December 1924 to February 1925. News and the general run of market comments at this time were not such as to indicate that anticipations for the coming season were dominant market factors, but study of differential price behavior does so indicate. The inference is that, wheat prices having recovered from their depression of July 1922 to May 1924, ideas on their "normal" level were in a state of flux.

2. Price movements dominated by changing appraisals of shortage of old-crop supplies occurred in February–May 1909 and in March 1925. In both cases, the chief price change occurred in the May future, but the new-crop futures were carried along with the old-crop futures, the July responding more than the September.[4]

3. A price movement dominated by a "squeeze" is clearly apparent in April–May 1908. Evidence of the character of the influence affecting the price of May wheat appears in the great widening of the price spread between May and July wheat, without corresponding widening of the spread between July and September. An abnormal market sitution of broadly similar character, but with notable special peculiarities, is reflected in price relationships in January–May 1926. May and cash wheat were held at a premium over new-crop futures that was abnormally large whether judged by the relation between July and September wheat or by the level of old-crop supplies of wheat.

ANALYSIS OF PRIMARY AND SECONDARY ELEMENTS IN PRICE MOVEMENTS

In study of differential price behavior as between North American markets and Liverpool, it has proved highly fruitful to go beyond comparison of gross price movements to a comparison of selected elements

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[4] In these large price movements, most suitable for graphic study, the new-crop futures followed the old-crop somewhat more closely than would be expected from the average ratios of change—100:41:12—given above. This seems probably attributable to concomitant developments bearing specifically on the next-season price outlook rather than to the existence of any general tendency for prices of new-crop futures to follow those of the old-crop more closely in large price movements of the latter than in smaller movements.
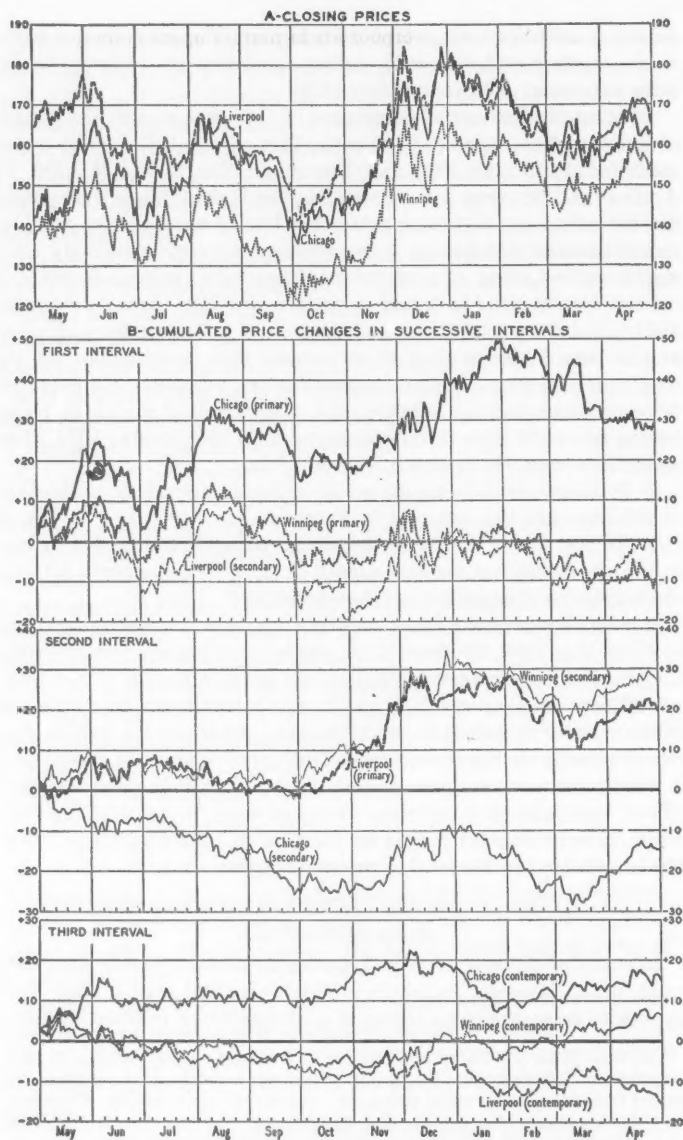
FIGURE 2.—Daily Wheat Prices and Cumulated Interval
Changes, 1925–26. See page 425.

in the movements. The Liverpool wheat market opens daily five hours ahead of the principal North American markets and closes shortly after the opening of their market sessions. If the Liverpool price has moved strongly up or down during its session, North American markets will almost always respond with a roughly corresponding price change recorded in their opening prices. Similarly, a strong price change in North American markets after the close of the Liverpool market is usually reflected in the opening price at Liverpool next day.

In the light of these tendencies, the gross price movement in each market over several days or weeks may be regarded as including a cumulation of *primary* movements made during its own session plus a cumulation of *secondary* movements made overnight in response to movements in other markets. There is also a small amount of *contemporary* movement during the brief interval in which Liverpool and North American markets are open simultaneously. In Figure 2 the gross price movements in Chicago, Winnipeg, and Liverpool, during May 1925–April 1926, shown in Section A, are analyzed into these three components in Section B.[5]

This analysis reveals that the price movements in all markets during May–September 1925 were dominated by movements that originated as primary changes in North American markets. The primary changes in Liverpool were small and irregular. In the sharp price rise of October–November, however, dominance lay with Liverpool. During these

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FIGURE 2

Prices and changes in U. S. cents per bushel. Based, for Chicago, on the September future, May 1–August 31, and on the May future September 1–April 30; for Winnipeg, on the October future, May 1–September 30, and on the May future, October 1–April 30; for Liverpool, on the October future, May 1–September 30, on the December future, October 1–October 28, and on the May future, October 29–April 30. Crosses mark changes in level of curves due to shifts from one future to another.

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[5] Separate cumulation of the individual primary, secondary, and contemporary price movements, respectively, is necessary to obtain clear graphic representation. The results of these cumulations, as they appear in the curves in Section B of Figure 2, have no individual counterparts in actual price statistics, but their sums, for each market, follow a course identical with that of the actual price movement in that market.

The substantial degree of parallelism among the curves of cumulated interval changes in the first two divisions of Section B illustrates the degree to which primary changes in the two North American markets tend to correspond and be followed by corresponding secondary changes in Liverpool, and the degree to which primary changes in Liverpool tend to be followed by corresponding secondary changes in the North American markets. There is a similar tendency for correspondence among the usually smaller contemporary changes shown in the final division of the chart.

months the net primary movement in Chicago was small, but in Liverpool the net primary movement was large.

Evidence on the primary origin of a price movement through such a differential price analysis may be highly illuminating in at least two respects. (1) It may resolve doubts as to the character of the influences responsible for the movement. Despite the confident tone of most contemporary market comments that ostensibly explain price movements, there is often ground for questioning the accepted interpretations. (2) It may throw light on the character of reaction to be expected from a particular movement. Because Liverpool is less sensitive and volatile than North American wheat markets, strong price movements initiated in Liverpool seem likely to be more soundly based in objective fact and less subject to severe reaction than price movements initiated chiefly in North America.

GENERAL CHARACTERISTICS OF CONCLUSIONS

Finally, a word should be said on certain general characteristics of the conclusions drawn from our analyses of differential price behavior, reflected in the foregoing assertions regarding tendencies of price behavior.

In the main, and perhaps in their entirety, these assertions are such as might have been made by an economic theorist who had given no close study to the behavior of wheat prices. Even more surely, they are assertions which might have been made by a business man relying on no critical quantitative analyses but on some years of contemporary observation of wheat price movements and on generalizations current in the trade. The assertions as here made are in fact founded on critical, and indeed very laborious, statistical analyses. In the original monographs presenting the results, there will be found numerous correlation and regression coefficients, calculations of standard errors, charts with fitted curves and regression surfaces. But in undertaking to present briefly the main results of the studies, I find it possible to neglect all these paraphernalia of quantitative statistical analysis and give the conclusions substantially in the form in which they might have been given by a pure theorist or by a business man. Is there then any merit in the laborious statistical origin of the conclusions?

The answer to this question may be suggested briefly. Ask a number of economic theorists reasonably familiar with the terminology and mechanism of the wheat markets to outline the tendencies which they would expect on primarily *a priori* grounds and you may obtain statements in agreement with all that has been said above. But you will get, I think, many more statements at variance with what has been said than in agreement. Probably you will find the theorists in substantial

agreement in supposing the existence of some tendencies which are contrary to those actually observed. The merit of the conclusions based on statistical analysis is not in their form or content, but in their superior trustworthiness.

Ask a number of intelligent business men of long experience with wheat prices to state the tendencies they believe to exist and you will get much more general agreement with the propositions here stated than will be obtained from economic theorists. Even among the business men, however, diversity of opinion will be found, and on some points the consensus will be contrary to conclusions derived from the statistical analysis. Again, the merit of the conclusions founded on the statistical analysis appears in a degree of superiority in the confidence with which they may be accepted.

Having been led in these final remarks to emphasize a secondary aspect of the main subject of this paper, may I offer a two-fold suggestion by way of summary?

1. The device of analyzing differential price behavior seems to hold great promise as a means of permitting statistical commodity price analyses to cope with the diversity and complexity of actual price phenomena; and

2. those of us who have faith in the advantages of quantitative analyses of economic phenomena may have concentrated our attention too much on its potentialities for yielding conclusions in quantitative form and have underrated its merits for obtaining qualitative conclusions of superior trustworthiness, where complexity of the problem or paucity of data set narrow limits to the quantitative conclusions.

HOLBROOK WORKING

Food Research Institute
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NOTE ON SHIFTS IN DEMAND AND SUPPLY CURVES

By Harry Pelle Hartkemeier

Descriptions of the shifting of demand and supply curves frequently are confusing. Some authors use the words *up* and *down*, others *vertical* and *horizontal*, while still a third group prefers *right* and *left*. Recently one writer[1] attempted to prove that a shift of the demand curve to the right graphically is the same as a shift upward, but he apparently failed to perceive that the demonstration was necessarily limited to the special case of a negatively sloping straight line.[2] Under this particular restriction, clearly

a shift upward ≡ a shift to the right,
a shift downward ≡ a shift to the left.

The analogous case of the positively sloping straight line supply curve[3] would be

a shift upward ≡ a shift to the left,
a shift downward ≡ a shift to the right.

In view of the possibility that confusion is likely to arise from these differences between the two curves, the retention of both terms, horizontal shifts and vertical shifts, seems justified.[4] It is proposed to show

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[1] F. L. Thomsen, " 'Vertical' and 'Horizontal' Shifts of Demand," *Journal of Farm Economics*, July 1933, pp. 567–70.

[2] It is peculiar that Thomsen should use only the straight line in his illustration since he stated the law of demand as, "The quantity taken varies inversely with the price" (p. 569). This statement translated into an equation would be $Q = 1/P$, which is a hyperbola when plotted on arithmetic paper.

[3] Thomsen does not discuss it, although he does discuss the negatively inclined supply-price curve.

[4] The principal, if not the only, purpose of Thomsen's note was to prove that a shift of the demand curve upward is the same thing as a shift to the right. However, this fact was pointed out long ago by Alfred Marshall in his *Principles of Economics* (8th edition) p. 97, for there we read, "When we say that a person's demand for anything increases, we mean that he will buy more of it than he would before at the same price, and that he will buy as much of it as before at a higher price. A general increase in his demand is an increase throughout the whole list of prices at which he is willing to purchase different amounts of it, and not merely that he is willing to buy more of it at the current prices." In a footnote on the same page referring to the statement just quoted we read, "We may sometimes find it convenient to speak of this as *a raising of his demand schedule*. Geometrically it is represented by raising his demand curve, or, *what comes to the same thing, moving it to the right*, with perhaps some modification of its shape" (*italics added in second place*). Marshall was careful to add "with perhaps some modification of its shape," a possibility which apparently escapes Thomsen. A change in shape would, of course, destroy Thomsen's proof that a shift upward is the same thing as a shift to the right.

that the statistical determination of demand and supply curves can be facilitated by considering these two kinds of shifts.

Figure 1A shows the demand curve DD plotted as a function of time (or, in other words, the demand plane giving the relationship between price, quantity, and time) under three different conditions. In plane ab the demand curve DD does not shift its position as time passes and in a two dimensional diagram it would appear as shown in
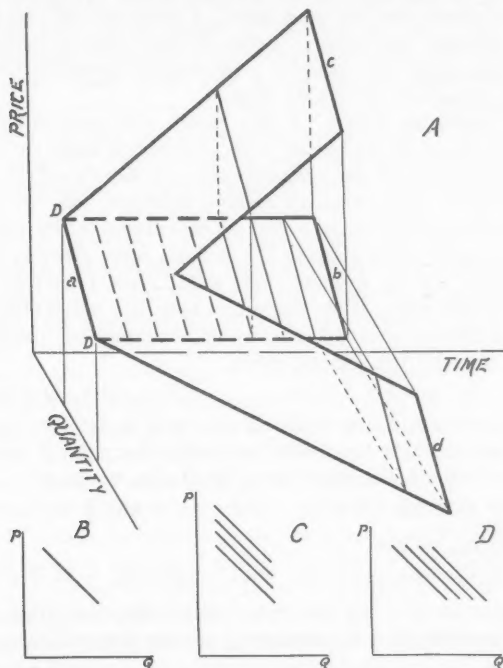
FIGURE 1—The shifting of the demand curve DD under three different conditions.

1B. In plane ac, the demand curve shifts its position upward as time passes, so that intersections of it with planes parallel to the price quantity plane (and perpendicular to the time axis) would appear as shown in 1C. In plane ad the demand curve shifts its position to the right as time passes, and Figure 1D shows how intersections of it with planes parallel to the price quantity plane would appear. There are advantages from a realistic or practical point of view in distinguishing between the two cases of a shifting demand curve, for plane ac indi-

cates that there is a positive trend in price but not in quantity, while plane *ad* indicates that there is a positive trend in quantity but not in price.[5] Plane *ac* indicates that the people consumed approximately the same quantities of the commodity at prices which gradually increased as time passed, while plane *ad* indicates that, at approximately the same prices, the people consumed quantities which gradually increased as time passed. From an economic point of view, these two situations are different and it is important to distinguish between them. To do so means that we must speak of both vertical and horizontal shifts of demand (and supply) curves. We need all four directions—up, down, right, and left—in order to distinguish properly between the different situations which may arise.

After examining Figure 1A, the reader who remembers his solid geometry will probably object to the distinction made by the author between plane *ac* and plane *ad*, since they are merely different parts or segments of the same plane or surface. However, the two planes are really one in only the special or limited case illustrated by the chart, in which the acute angle between the demand curve *DD* and the *Q* axis is 45 degrees and the demand curve shifts upward the same distance it shifts to the right. If the angle had been any value other than 45 degrees, a different demand curve would have resulted. Take the equation of a straight line demand curve,

$$aP + bQ + c = 0. \qquad (P = \text{price}, \ Q = \text{quantity}).$$

Adding a constant *k* first to *P* and then to *Q* to get the two demand curves, one of which has shifted vertically along the *P* axis and the other has shifted horizontally along the *Q* axis, we secure

$$a(P + k) + bQ + c = 0, \quad \text{and} \quad aP + b(Q + k) + c = 0.$$

Simplifying, we get

$$aP + bQ + ak + c = 0, \quad \text{and} \quad aP + bQ + bk + c = 0.$$

It is easy to see that the two equations are identical when $ak = bk$ or $a = b$. This means that the negatively sloping demand curve must be

[5] The reader may easily verify these statements by plotting one observation on each demand curve in plane *ab* (scattering them in a random manner) and projecting them vertically into plane *ac* and then horizontally into the price-time plane. This will result in an ordinary time series chart in which a positive trend in price will be noticeable. The reader should then return to the original observations in plane *ab* and project them horizontally into plane *ad* and then vertically (down) into the quantity-time plane. A time series chart will result and the reader will notice a positive trend in quantity. Likewise, if the original observations in plane *ab* are projected horizontally into the price-time plane and vertically into the quantity-time plane, the two time series which result will have no positive trends (the trends will be horizontal straight lines if the original observations were scattered in a random manner in plane *ab*).

inclined at an angle of 45 degrees in order to result in the same demand curve whether it shifts upward or to the right the same distance.

In actual practice, the vertical and horizontal shifts both occur together, and there is much to be gained in separating them rather than combining them or considering them identical. This is especially true when the two shifts are in such directions and amounts as to counteract each other. For example, suppose that there is a positive trend in price and a negative trend in quantity. The positive trend in price would cause the demand curve to shift upward, while the negative trend in quantity would cause the demand curve to shift to the left.[6]

Thomsen has pointed out that in the case of the demand curve, price is the independent variable and quantity is the dependent variable; while in the case of the supply-price curve (which is based upon another statement of the law of demand), price is the dependent variable and quantity is the independent variable. To secure the regression of quantity on price (the demand curve), we minimize the horizontal deviations and to secure the regression of price on quantity we minimize the vertical deviations, so Thomsen concludes that we should use the same direction to describe the shifting that we use to minimize the deviations. This is the horizontal direction in the case of the demand curve, and Thomsen decides to use the horizontal direction in the case of the supply-price curve also, "since the qualitative distinction between demand and supply-price curves generally is merely a matter of intellectual preciseness, and since any shift of either curve is the result of a shift in demand, the practice of referring to shifts of the curve (or of demand) as being to the right or left appears justifiable."[7] It seems strange that the difference between the demand curve and the supply-price curve is clearly pointed out and used as a basis for deciding that we should speak of horizontal shifts rather than vertical shifts, only to have the distinction depreciated and almost wiped out just as soon as the decision has been made.

Furthermore, it is erroneous to conclude that there is any necessary connection between horizontal and vertical shifting, and independent and dependent variables. Thomsen implicitly assumes that as soon as one has decided which of the two variables is the independent variable it is necessary to minimize the deviations between the observed and computed values in the direction perpendicular to the axis on which the independent variable is plotted (which means minimizing the deviations in the direction parallel to the axis of the dependent variable),

[6] For a discussion of vertical and horizontal shifting of the supply curve, see H. P. Hartkemeier, "The Supply Function for Agricultural Commodities," *University of Missouri Studies*, October 1, 1932. Chapter 3.

[7] F. L. Thomsen, *op. cit.*, p. 570.

but such is not the case. The deviations may be minimized in any direction—even in the direction perpendicular to the line which is to be fitted (the line of mutual regression). The direction in which we minimize the deviations between observed and computed values has nothing to do with the directions we should use to describe the shifting of the demand (or supply) curve. Even after it has been decided that price is the dependent variable and quantity the independent, it may be necessary to minimize the deviations in the horizontal direction if the shifting has taken place in this direction (shown in Figure 1D) and in the vertical direction if the shifting has taken place in this direction (shown in Figure 1C). The statement in the last sentence assumes that simple correlation of price and quantity is being used to secure the demand curve. This is not a very satisfactory procedure; multiple correlation is much better when they are really more than two variables, but when the shifting of the demand curve due to the passing of time has not been very great, simple correlation may give satisfactory results if the deviations are minimized in the direction of the shifting. And it may be stated again that we need all four directions to describe the shifting rather than just two; vertical directions must be used as well as horizontal.

The proof that a vertical shift of the demand curve is the same as a horizontal shift (which is limited to the case where the demand curve is a negatively sloping straight line) depends upon the extension of the demand curve beyond the range of actual observations. To defend such a practice, Thomsen must define the demand curve (the one determined from actual data) so as to include all possible extensions of the curve. This is certainly a questionable practice, for the standard error of the projected curve does not remain constant[8] (as does the standard error of estimate within the range of observations). The shifting of the curve to the right would not have to be very great before the standard error of the projection of the curve would be so large as to make it impossible to say that it coincided with the demand curve which had shifted upward. It is a serious error to confuse the theoretical demand curve and the one determined from actual data. Even in the case of the theoretical curve, it is necessary to distinguish between the neo-classical demand curve and the demand curve of the mathematical school.[9]

[8] Henry Schultz, "The Standard Error of a Forecast from a Curve," *Journal of the American Statistical Association*, June, 1930.

[9] The same distinction must be made for the supply curve. See H. P. Hartkemeier, *op. cit.*, Chapter 1 and Henry Schultz, "Der Sinn der statistischen Nachfragekurven," *Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung*, Heft 10, 1930.

Much of the seeming difference between the men referred to above arises out of a fairly simple difference or confusion in the definition of the concepts or terms used. There is nothing new in this different use of the term *demand* by two groups of economists. English economists meant the demand schedule or the whole demand function when they used the word *demand*, while many of the continental economists meant the quantity demanded when they used the word *demand*. To illustrate, Cassel wrote, "What we want to know is how the demand for an article changes when the price is slightly altered, all other prices remaining constant."[10] In contrast, Marshall could write, "What we want to know is how the quantity demanded changes when the price is slightly altered. . . ."[11] To Cassell *demand* means the quantity demanded, while to Marshall *demand* means the functional relationship between the quantity demanded and the price, so when the price changes slightly Cassel would say that demand changes, but Marshall would say that demand does not change. To tie this up with the shifting of the demand and supply curves, when one person says that there is a shift in demand he means that the demand function or whole curve shifts, while another person may say that there is a shift in demand to mean merely a shift to another point on the same curve or demand function. The use of the term demand (and supply) in two different senses would not be so unfortunate if each author would adopt one meaning and use it consistently throughout his book or article, but such is not the case. The confusion has extended to some writers so that the two meanings may be found in the same article, paragraph, and even the same sentence. "For example, if supply increases the supply curve will shift to the right, now being represented by. . . ."[12] Here, both terms, *supply* and *supply curve*, are used; what does the first term, *supply*, mean? One would think that it must mean something different from *supply curve*, since the latter expression is used in the same sentence, and if *supply* means *supply schedule* or *supply curve* the sentence doesn't make much sense—the last part merely repeating the first part (the conclusion is identical with the assumption). If the supply curve shifts to the right it shifts to the right! Perhaps the first word *supply* means *quantity supplied* rather than the supply function, and the reader is supposed to understand from the statement that ". . . if the quantity supplied increases, the supply curve will shift to the right. . . ." However, this statement is not true—at least, it is not a general truth—for if the quantity supplied increases, the supply curve may remain fixed and the price change. Some readers may ask how it is possible for the

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[10] Gustav Cassel, *The Theory of Social Economy*, p. 80.

[11] See footnote 4.

[12] F. L. Thomsen, *op. cit.*, p. 568.

quantity supplied to increase if the supply curve remains fixed, since the supply curve indicates only one quantity for each price. This is possible because the quantity supplied is hardly ever the exact quantity which will result in the equilibrium price; the quantity supplied varies decidedly from year to year. (If it were not for this variation it would be more difficult to determine the demand and supply curves.) Besides, the supply curve shows what the producers are willing to produce at each price—not what they actually produce. In the field of agriculture, weather factors partly determine the actual production,[13] which is seldom equal to the intended production which would be expected on the basis of the supply curve. Even in the industrial field numerous factors can prevent the manufacturer from supplying exactly what he would like to supply. Raw material may not be available or may be delivered late, machinery may break down, labor difficulties may interfere with the manufacturer's plans.

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[13] For a statistical measurement of the effect of temperature on the production of potatoes and eorn, see H. P. Hartkemeier, *op. cit.*, Chapters 3 and 4.

THE STATISTICAL ANALYSIS OF STOCK PRICES

By Victor S. von Szeliski

THE purpose of this paper is to lay the groundwork for statistical methods of studying technical market action so-called, which is now carried on almost wholly by "chart reading" (for instance, as in *Stock Market Theory and Practice*, by Schabacker).

STOCK MARKET
Daily Price and Volume
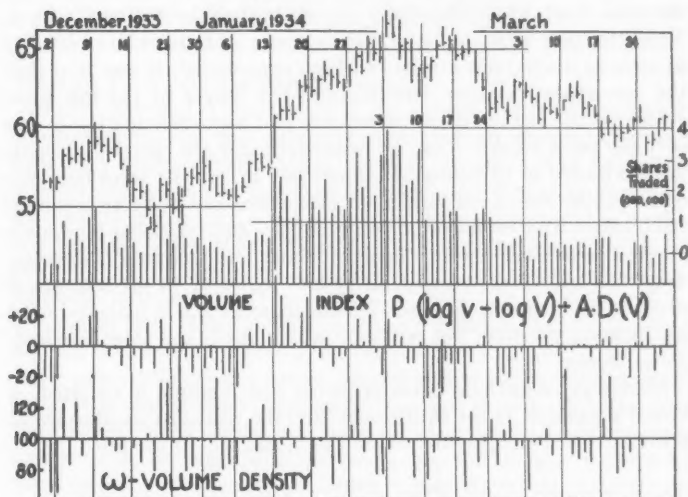
Newspapers furnish the raw statistical material: volume of trading during the day or other interval, and the price movement, high, low, and close. This raw material is obviously not usable as given; it must be worked up into coefficients or indexes of some sort. The chief subject of this paper will be methods of transforming prices and volumes so as to get as many possible comparable repetitions, and permit a count of favorable and unfavorable cases.

I. PRICE

Prices and price changes as quoted daily in the newspapers in points are not technically comparable. Not only are price changes as measured in points not comparable for the same stock at different price levels—

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a point move in Anaconda selling at 7 is technically more significant and statistically less probable than a point move when it is selling at 160—but the prices of different stocks at any one time are not comparable. The point moves of Anaconda cannot be compared with the point moves of American Telephone. Unless we invent some way of remeasuring these prices in terms of some common denominator, we forego the possibility of getting large numbers of essentially repeated observations.

Let us call the price of the stock p, and let us call p' the to-be-discovered transformed price, such that equal changes in p' have an equal technical significance for different stocks or for the same stock at different times. The first answer to this problem was furnished by the ratio chart, which essentially consists in the substitution $p' = log\ p$. This was fairly satisfactory as long as stocks sold above 30 or 40. But as soon as lower price ranges had been experienced, it was seen that the percentage measure overestimated the moves of the low price stocks, as the arithmetic point measure had overestimated the moves of high price stocks. That is, whereas under the point-arithmetic system moves of 10 points in a single day in two- or three-hundred-dollar stocks are of not infrequent occurrence and of lesser technical significance, and moves of 10 points in five- and ten-dollar stocks are most unusual and highly important from the technical viewpoint, under the percentage-logarithmic system moves of 10 per cent are of common occurrence in five- and ten-dollar stocks and of little technical significance, yet occur but rarely in two-hundred- and three-hundred-dollar stocks.

Obviously, something between points and percents is required. A recent suggestion is the square-root law: the technical significance of price movements is proportional to the square root of the price,

$$\Delta p' = \text{const. } \Delta(p^{\frac{1}{2}})$$

and in the limit,

$$dp' = \text{const. } d(p^{\frac{1}{2}});$$

$$p' = \text{const. } p^{\frac{1}{2}}.$$

Thus, a move in a low-priced stock from 4 to 9 is technically equivalent to a move in a high-priced stock from 144 to 169 (because the square root of 9 minus the square root of 4 is equal to the square root of 169 minus the square root of 144).

Two points are to be noted:

1. The question is one of transforming price *changes*, Δp, into technically significant price changes, $\Delta p'$, and not prices, except in so far as the computation of the transformed price p' may facilitate the calculation of $\Delta p'$.

2. Each of these three laws, point, percentage, and square root involves a power of p: p^0, p^1, $p^{1/2}$. The point measure is

$$dp' = \frac{dp}{p^0} = dp\,;$$

the percent measure is

$$dp' = \frac{dp}{p} = C_1 d \log p\,;$$

and the square root measure is

$$dp' = C_2 d(p^{\frac{1}{2}}) = C_3 \frac{dp}{p^{\frac{1}{2}}}\,;$$

where C_1, C_2, C_3, are appropriate constants.

Which exponent is correct, the 0, the $\frac{1}{2}$, or the 1? Is there any way of determining the best exponent directly from observations? One possibility suggests itself, that is, study of the one- or two-day moves of individual stocks as functions of their price. In the short market movements of less than a week, technical forces dominate the market and the price effect should therefore show up. The longer trends are wholly a matter of economics and, therefore, cannot be regarded as following any analytic formula at all. There is no justification for attempting to fit the longer moves into any mathematical strait-jacket, and the fact that the 1929–32 bear market, as measured by the Dow-Jones industrials, plots between parallel straight lines on square-root paper is no more a proof of the square-root law than the fact that prior to 1929 other stock indexes fluctuated between parallel upper and lower straight lines on ratio paper proves the percentage law. We can observe response to technical situations only in the initial phase of long moves and when the whole market reacts to surprise news. Under such conditions we may assume that the high-low range is technically the same at all price levels. The inevitable chance variations in technical equivalence of movement as between stock and stock can be overcome by averaging and graduation. As an illustration of what we mean, we present the following table showing the high-low range of approximately 130 active stocks on June 6 and 7 of 1930, and February 10, 1931, grouped according to price.

The observations were divided into thirteen groups, each containing about ten cases, group 1 being the ten lowest in price, group 2 the next ten, etc. Group 1 had a median price of 21 and a median range of 1 point. At the other end of the scale, Group 13 had a median price of 232 and a median range of 4.7 points. If these observations are plotted

TABLE I

HIGH-LOW RANGE AS A FUNCTION OF PRICE

Group	Observations from June 6 and 7, 1930		Observations from Feb. 10, 1931	
	Median price, P	Median range, R	Median price, P	Median range, R
1	21	1.0 point	10	4
2	27½	1.0	15	6
3	32½	1.5	19	9
4	42½	1.6	26	7
5	48½	1.6	34	10
6	59½	1.7	45	13
7	72	2.5	56	121
8	81	2.4	68	16
9	89½	4.2	93	38
10	103	3.6	145	36
11	114	3.1		
12	151	5.5		
13	232	4.7		

Note: $\log R = 0.77 \log P + \text{const.}$ $\log R = 0.88 \log P + \text{const.}$
        or
        $R = c \cdot p^{0.77}$
Active stocks only.
About 10 stocks in each group
Logarithms to base 10.

on double log paper, the slope of the best fit straight line will indicate the best exponent for describing the relation between $\Delta p'$ and p. For the above example the slope is 0.77, that is, the high-low range of prices varies approximately as the ¾ power of the price. Neither the square root rule, exponent ½, or the percentage rule exponent, 1, is supported by these observations, but rather a law about halfway between. For other cases exponents of 0.95 and 0.88 were found. The average slope for the three cases was 0.86. This strengthens the view that the technically significant measure of price change is something between points and per cent and not the square root. As a result of this study an exponent of 0.75[1] was selected somewhat arbitrarily, rather than the

[1] The exponent 0.75, being based on only a few observations, can be accepted only on probation. It is useful enough in practice, however, to demonstrate its superiority over the percentage and square root laws. It would be highly desirable to study price changes on, say, 100 days selected from the markets of 1928–1934 as a function of volume of trading and price level:

$$\Delta p = C \cdot f(v, p),$$

and measure the variance at different price levels and under various conditions (v in the above equation probably ought to be measured by value of shares traded, or fraction of total capitalization traded, rather than number of shares). $\partial f / \partial p$ will then be the desired law.

average observed exponent of 0.88. Let us call the new price measure "isopoints," the "iso-" being adopted to indicate technical equivalence. Price changes of different stocks may be compared by dividing each change by the $\frac{3}{4}$ power of the price level at which each occurs.

$$\Delta p' = Cdp/p^{\frac{3}{4}}.$$

It will be convenient to determine C so that a price change from 100 to $101 = 1$ isopoint. C is found to be $1/100^{\frac{3}{4}}_4$, or 0.031623. The following table shows the significance of a price change of 1 point at various price levels in terms of the new measure.

TABLE II
TECHNICAL SIGNIFICANCE OF PRICE CHANGES
*Significance of a Price Change of One Point in Isopoints, and Arithmetic
Points Equivalent to 1 Isopoint, at Various Price Levels*

Price level	Change of 1 arithmetic point expressed in isopoints	Change of 1 isopoint expressed in points
10	5.623	0.178
15	4.150	0.241
20	3.343	0.299
30	2.467	0.405
40	1.988	0.503
50	1.682	0.595
75	1.241	0.806
100	1.0	1.0
125	0.846	1.18
150	0.738	1.355
200	0.595	1.68
300	0.439	2.28

A decline from 150 to 141 is $9 \times 0.738 = 6.65$ isopoints
A rally from 200 to $210\frac{1}{2}$ is $10.5 \times 0.595 = 6.225$ isopoints

One arithmetic point expressed in terms of isopoints in shown in Table II.

In summary, the fundamental assumptions in this method of determining the technical significance of prices is that, in the initial stages of important moves, technical factors predominate, and that the high-low range is almost wholly a reflection of technical factors. The law connecting price range and price is found to approximate the $\frac{3}{4}$ power of the price.

From the relation

$$dp' = \frac{dp}{\dfrac{1}{100^{\frac{3}{4}}} \cdot p^{\frac{3}{4}}},$$

TABLE III.

MARKET AVERAGES AND TOTAL VOLUME
DECEMBER, 1933–MARCH, 1934

		Volume (000)	High	Low	Close	ρ	v/F_1	ω_1	F_1
Dec.	1*	814	57.39	56.38	56.62	-20	79	86	
	2–4	671	56.88	56.09	56.54	-23	64	66	1045
	5	2,010	58.65	56.70	58.31	$+24$	126	123	1590
	6	1,443	58.78	57.71	58.07	$+ 9$	93	84	1555
	7	1,693	59.13	57.84	58.40	$+15$	142	124	1190
	8	1,330	58.83	57.57	58.04	$+ 4$	108	90	1230
	9	1,070	59.17	57.85	59.05	$+20$			
	11	2,453	60.30	58.65	59.13	$+26$	138	115	1770
	12	1,658	59.77	58.48	58.90	$+ 4$	127	107	1300
	13	1,333	59.33	57.98	58.45	$- 6$	108	96	1240
	14	1,556	59.67	58.19	58.87	$- 1$	98	93	1595
	15	1,173	58.80	57.41	57.75	-12	96	94	1220
	16		57.64	56.23	56.62	$+ 6$			
	18	1,348	56.81	55.31	56.11	$+ 5$	94	95	1445
	19	1,026	56.57	55.51	55.98	-14	86	86	1185
	20	2,146	56.43	53.91	54.84	$+16$	105	103	2040
	21	1,021	55.32	54.38	54.77	-14	83	80	1235
	22–23	2,104	56.83	55.33	56.07	$+33$	143	136	1475
	26	1,304	55.91	54.52	54.93	$- 4$	102	98	1280
	27	3,075	56.30	54.29	55.40	$+28$	172	168	1790
	28	1,478	57.47	55.66	57.04	$- 7$	80	80	1845
	29	1,124	57.44	56.43	56.79	-17	78	83	1450
Jan. Feb.	30 2 }	1,346	58.57	56.49	57.61	$- 9$	86	99	1575
	3	1,383	58.01	56.04	56.82	$- 7$	80	98	1740
	4	1,188	56.88	55.65	55.85	-15	111	139	1067
	5	1,055	57.06	55.65	55.85	-18	65	81	1620
	6– 8	785	56.37	55.41	55.68	-25	73	85	1070
	9	860	56.85	55.89	56.27	-14	90	97	950
	10	1,416	57.95	56.49	57.82	$+10$	113	113	1250
	11	1,600	58.62	57.32	58.09	$+15$	101	93	1580
	12	1,550	58.43	57.30	57.56	$+12$	147	122	1055
	13	750	57.83	57.03	57.53	$+ 7$			
	15	3,743	60.85	58.16	60.60	$+45$	107	82	3495
	16	3,445	61.90	60.45	61.10	$+33$	140	97	2460
	17	2,850	62.01	60.46	61.11	$+16$	169	107	1685
	18	2,126	61.76	60.47	60.94	$+ 2$	159	95	1335
	19	3,542	62.97	61.12	62.60	$+22$	200	113	1770
	20	1,954	63.30	62.30	62.59	$+32$			
	22	2,663	63.45	61.90	62.38	$- 1$	154	83	1735
	23	2,384	63.27	61.78	63.05	$- 8$	206	100	1160
	24	3,358	64.23	62.94	63.46	$+ 6$	271	122	1240
	25	2,270	63.72	62.42	63.21	-13	171	72	1330
	26	2,510	63.75	62.53	62.87	$- 6$	213	85	1180
	27	1,200	63.10	62.31	62.63	$- 6$			
	29	2,784	64.06	62.63	63.71	$+ 1$	278	109	1000
	30	4,237	65.19	63.79	64.55	$+18$	334	132	1270
	31	3,150	64.91	63.41	63.66	$+ 3$	232	92	1360

TABLE III, *Continued*

		Volume (000)	High	Low	Close	ρ	v/F_1	ω_1	F_1
Feb.	1	4,714	65.89	64.18	65.41	+21	271	111	740
	2–3	3,300	65.89	64.18	64.51	+ 2	180	78	1835
	5	4,941	67.48	65.76	66.73	+18	195	94	2535
	6	4,330	67.36	65.75	65.92	+ 8	215	112	2010
	7	4,500	66.98	64.35	65.25	+ 7	202	113	2230
	8	3,200	65.70	63.79	65.46	−11	175	98	1830
	9	3,337	65.68	62.91	64.05	−11	140	76	2375
	10	2,187	63.88	62.05	63.74	+ 3			
	13	2,060	64.57	63.15	63.88	−33	142	75	1450
	14	1,939	64.54	63.01	64.31	−30	178	92	1090
	15	2,977	65.61	64.45	65.29	−27	314	165	950
	16	2,769	66.52	65.30	65.46	− 9	185	100	1495
	17	1,165	65.92	64.98	65.60	−11			
	19	2,347	65.89	64.33	64.64	−11	183	100	1290
	20	1,220	65.26	64.37	64.89	−33	121	70	1010
	21	1,899	65.63	64.72	65.08	−10	118	118	1010
	23	2,290	64.95	62.84	63.50	+ 1	117	86	1955
	24	1,223	63.49	62.43	62.57	+ 7			
	26	2,188	62.28	60.80	61.22	+ 1	132	113	1650
	27	1,272	62.62	61.05	61.71	−25	86	84	1490
	28	1,322	62.72	61.29	61.32	−18	96	106	1375
Mar.	1	1,242	61.72	60.25	60.78	−20	95	112	1310
	2–3	1,520	63.10	61.62	62.83	− 6	82	96	1860
	5	952	62.95	61.92	62.39	−24	60	68	1585
	6	806	62.49	61.62	61.89	−27	94	98	883
	7	1,734	62.32	60.26	60.52	+ 8	101	97	1725
	8	1,697	61.65	60.01	61.30	+ 8	120	106	1420
	9	1,370	61.98	60.87	61.06	+ 1	110	90	1252
	10	572	61.32	60.61	60.89	− 6			
	12	1,259	61.85	61.15	61.82	− 2	185	145	680
	13	1,275	62.89	61.80	62.31	− 2	119	91	1070
	14	1,359	62.98	61.98	62.19	+ 2	115	89	1180
	15	1,342	62.19	60.79	61.28	+ 2	116	92	1160
	16	1,169	61.77	60.75	61.47	− 6	102	86	1140
	17	725	61.55	60.66	60.87	+ 2			
	19	1,513	60.49	59.40	59.65	+ 5	123	113	1230
	20	1,544	60.85	59.56	60.44	+ 6	144	140	1070
	21	1,068	60.28	58.91	59.54	− 9	77	78	1385
	22	1,054	60.46	59.18	59.74	− 9	79	82	1340
	23	792	60.34	59.66	59.85	−20	90	94	880
	24	681	60.75	59.84	60.50	+ 3			
	26	1,275	61.53	60.35	60.49	+ 3	94	96	1360
	27	1,590	59.49	58.34	59.12	+10			
	27–28	1,215	59.49	58.34	59.31	0			
	29	1,020	60.64	59.44	60.60	− 6			
	31	814	60.88	59.81	60.73	+12			

Stock averages are Dow-Jones averages for industrials, rails and utilities, weighted 2, 1, and 2, respectively.

Whenever the high-low range of one day lies within the range of the preceding or succeeding day, the two have been combined, using the wider range and the close of the second day. The volumes of the two days are averaged.

ρ, ω, and F, have not been calculated for Saturdays.

p' can be deduced by integration;

$$p' = \tfrac{1}{4} \times 100\tfrac{3}{4} \times p^{\tfrac{1}{4}}.$$

p' may be called the isoprice. A table for $10 \leqq p \leqq 110$ is given, which is used just like a logarithm table. Isopoint changes are given directly by subtraction. Thus, the measure in isopoints of a price change from 30 to 31 is given by $298.5 - 296.0 = 2.5$ isopoints.

The best daily stock price index, the index of 90 stocks published by the Standard Statistics Company, cannot be converted into isopoints directly because the average value of the stocks composing it is much less than the numerical value of the index. Since September, 1930, the multiplier to reduce to average prices has been about 0.37.

II. VOLUME

How can we compare the volume of trading on the New York Stock Exchange during the summer of 1934, when the average volume was perhaps 600,000 shares a day for long periods at a time, with the conditions of 1929, when the average volume of trading was above 5,000,-000 shares for months?

How can we compare the volume of a stock like Kroger Grocery, which may average only 2,000 shares a day, with the volume of a stock like United Aircraft, which may average 50,000 shares a day? The first thing that suggests itself is to consider each day's volume, v, in relation to the average for some period, which average we may call the base volume and denote by V. In the present study, the period chosen was the nine days immediately preceding the day whose volume was to be measured.[2] The ratio of the volume on a given day (denoted by v) to its proper base volume V, may be called the volume ratio, designated by U. Uniformities then begin to appear. Genuine moves usually start with volumes at least equal to the base volume and as a rule 10 or 20 percent larger. The volume on secondary reactions is generally much less than the base volume, and it may diminish to 60 or 70 percent of V. Thus, it is possible to replace the vague, qualitative phrases, "breaking through on volume," and "drying up," by quantitative ratios such as 110 or 60 percent.

This still leaves something to be desired. A difficulty analogous to the difficulty of comparing high-priced and low-priced stocks appears. At one extreme, there is the market as a whole in which a volume de-

[2] It is desirable to have as long a period as possible for a base. But market conditions change so quickly that a base stretching much farther back than ten days frequently is obviously too large or too small against which to measure current volume. The nine-day period was selected after study of a number of stock charts. If anything, it is too long. A nine-day base has also been used for the total trading on the New York Stock Exchange.

viation of 25 percent from the base is rather unusual; at the other, inactive stocks such as Kroger Grocery and Air Reduction in which volume deviations as much as 100 percent above the base are by no means infrequent. If we make the assumption that equi-probable volumes have equal technical significance, we are led to study individual volumes by comparison with the frequency distribution of volumes in which they occur.

TABLE IV
FREQUENCY DISTRIBUTIONS OF W

Full lots W	Class intervals x	First group			Second group		
		Actual	x'	Calculated*	Actual	x'	Calculated
0	0	⎧ 0	0.0	4.17	⎧ 0	0.0	2.84
15	1	30⎨ 2	0.637	11.4	10⎨ 0	0.588	6.97
30	2	⎩28	1.274	21.7	⎩10	1.175	13.7
45	3	37	1.911	33.2	29	1.764	20.9
60	4	49	2.548	42.7	28	2.352	27.4
75	5	62	3.185	48.2	35	2,940	31.8
90	6	43	3.822	48.7	35	3.528	33.5
105	7	44	4.459	44.8	36	4.116	32.4
120	8	33	5.096	38.0	28	4.704	29.0
135	9	25	5.733	29.9	16	5.292	24.3
150	10	27	6.370	22.1	17	5.880	19.2
165	11	7	7.007	15.4	13	6.468	14.4
180	12	8	7.644	10.1	7	7.056	10.2
195	13	9	8.281	6.4	5	7.644	7.0
210	14	5	8.918	3.8	8	8.232	4.6
225	15	⎧2	9.555	2.1	⎧3	8.820	2.85
240	16	⎪0	10.192	1.2	⎪1	9.408	1.72
255	17	⎪2	10.829	.63	⎪1	9.996	1.00
270	18	⎪0	11.466	.35	⎪1	10.584	.57
285	19	6⎨1	12.103	.16	5⎨0	11.172	.31
300	20	⎪0	12.740	.08	⎪0	11.760	.16
315	21	⎪1	13.377	.04	⎪1	12.348	.09
330	22	⎪0	—	—	⎪0	12.936	.04
345	23	⎩0	—	—	⎩1	13.524	.02
Total—N		385			285		

W is, approximately, the number of full lots recorded when the expectation is 100 full lots per day. Frequencies are calculated by the equation,

$$F(x') = \frac{Am^{x'}}{\Gamma(x'+1)}.$$

*The calculated frequencies are ordinates of $F(x')$, not areas. A more exact treatment would, of course, require the use of areas.

Table IV shows the frequency distribution of 385 ratios W of v to the centered average volume V' That is, W is the ratio of the daily volume to the nine days' moving average, centered. These ratios

should not be confused with the ratios $U = v/V$, where V is calculated from the nine preceding days. For the purpose of studying the frequency distribution of the volumes, it is better to use V centered instead of V trailing,[3] although in actual market analysis we have to use the latter. The W's were calculated from the volume records of eighteen trading favorites during 1930–32, and were restricted to those days for which $9,000 \geqq V' \geqq 11,000$. As the variance of the distribution, and probably the higher characteristics also, certainly varies with V, the observations should include only W's for which the V's do not vary widely. Otherwise, we run the risk of having a compound frequency distribution, and so making an erroneous estimate of the variance. A few observations for which V was changing very rapidly were omitted. The ratios have been multiplied by 100, so that the table represents, substantially, the number of full lots recorded[4] when $V' = 100$.

These frequencies may follow the Poisson law. They are analogous to, say, the number of telephone calls going through an exchange in successive intervals of time. These latter follow the law[5]

$$p_0 = e^{-kt},$$
$$p_1 = kte^{-kt},$$
$$p_2 = \frac{k^2t^2}{2} e^{-kt},$$

etc., or, in general,

$$p_n = e^{-kt} \cdot \frac{k^n t^n}{n!}.$$

But there is a very important difference. Whereas a given telephone call is relatively independent of the call frequency in the immediately preceding interval—the subscriber in general being ignorant of the frequency—a given stock transaction is often directly due to the pre-

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[3] It would probably be still better to use the V extending five daily volumes back from v, and forward three daily volumes, because volume surges often arrive explosively, although they always tend to die out exponentially. By using V', we run the risk of comparing v to a graduated volume dependent on undiscountable events in the future. Graduation assumes that the "true" values of the variables are smooth and due to continuously varying underlying causes, but such events as the Glass-Steagall Bill in February, 1932, or the Hoover Moratorium in June, 1931, constitute discontinuities.

[4] Certain types of full-lot transactions are not recorded. Odd-lot transactions, which may take up a very large portion of the total trading, are usually not recorded, although several may be bunched and get recorded as full lots.

[5] See Thornton C. Fry, *Elementary Differential Equations*, p. 139; *Probability and Its Engineering Uses*, pp. 226 ff.

ceding activity and price change, a tendency for which the chart readers, board room traders, and floor traders are responsible. Share transactions are not distributed at random.

A glance at the distribution shows that it is skew, as might be expected from the fact that it is limited in one direction and unlimited in the other. The characteristics of the distribution are shown in Table V.

TABLE V

CHARACTERISTICS OF THE FREQUENCY DISTRIBUTIONS OF W

	First group		Second group		Combined groups in class intervals
	In class intervals	In full lot units	In class intervals	In full lot units	
N	385	385	285	285	670
k_1	6.40	9.60	6.94	104.1	6.73
k_2	10.12	2,280	11.88	2673	10.90
k_2'	10.08	2,268	11.80	2655	
σ	3.17	46.6	3.43	51.4	3.29
k_3	32.65	110,000	48.73	165,000	39.35
k_4	157		303		
g_1	1.02		1.24		1.09
a	1.569	23.53	1.70	25.50	3.60*
m	4.875		4.08		
χ^2	16.9		23.3		
P	0.13		.04		
b					1.02

For description of the k-characteristics see R. A. Fisher, *Statistical Methods for Research Workers*, Fourth Edition, p. 74. k_1 is the mean, $k_2 = \sigma^2$, the variance.

* 54.0 full-lots.

It is rather surprising that the mean number of full lots recorded is 96.0 instead of 100. The explanation is, probably, that the centered moving average, V', is a poor graduation value when V is changing rapidly; also, that the general volume of trading during the period under review, 1930–32, was very erratic, and that the rejection of observations for which $V' < 90$ and $V' > 110$ may have biased the result in some way.

The full lots recorded clearly are not the Poisson distribution, for $k_1 \neq k_2 \neq k_3 \neq k_4$. The full lot is not the proper unit for counting the frequency of independent transactions.[6] Thus, if a trader decides to go long of 100 GIL, and he buys 100 @14$\frac{7}{8}$, 500 @ 15, 100 @ 15, 200 @ 15$\frac{1}{8}$, and 100 @ 15$\frac{1}{4}$ this is clearly only one decision and one independent transaction, although ten full lots and three prices occur.

[6] See Fry, *Probability and Its Engineering Uses*, p. 218. "A set of points is said to be distributed 'collectively at random' along a line segment provided the probability of any interval dx containing n points is independent of the number of points in any interval not wholly or partly in dx."

If we replace W by $x' = W/a$, where $a =$ the average number of full lots per independent transaction, the Poisson curve may fit. a should be chosen[7] so that $k_1(x') = k_2(x')$, i.e., so that the mean of the transformed distribution is equal to the square of the standard deviation.

$$k_1(W/a) = \frac{1}{a} k_1(W) = k_2(W/a) = \frac{1}{a^2} k_2(W),$$

$$k_1(W) = \frac{1}{a} k_2(W),$$

or

$$a = \frac{k_2(W)}{k_1(W)},$$

which gives $a = 25.5$ full lots. Table IV shows the new abscissa x', and the frequencies calculated from the law,

$$F(x') = \frac{A m^{x'}}{\Gamma(x' + 1)},$$

where $m = 4.075$. $\chi^2 = 16.9$, and $P = 0.13$, indicating a tolerable fit.

This means that in a full days' trading of 10,000 shares recorded there are only four independent transactions, which is somewhat of a strain on the imagination.

Another frequency distribution was made of the W's of another group of trading favorites for the period 1930–32, this time more care being taken to reject unusual cases, especially where extraordinary volume surges were taking place. The requirements were,

> V' for the day in question between 9,000 and 11,000,
> V' for the preceding day also between 9,000 and 11,000,
> V' for succeeding day $> 9,000$.

This frequency distribution and its characteristics are shown in Tables IV and V. a is 23.3 full lots, $m = 4.08$. The fit is definitely poor, $\chi^2 = 23.3$ and $P = .04$.

As might be expected from the oscillatory character of W, the distributions tend to be flat-topped.[8]

A more prominent characteristic of the observed frequency distributions is the large deficit at the lower ends. The frequency in the lowest two class intervals of the first distribution is 2, against 15.57 expected; in the same class intervals of the second distribution, zero, against 9.81 expected.

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[7] Cf. Arne Fisher, *Mathematical Theory of Probabilities*, p. 275.

[8] See Dickson H. Leavens, "Frequency Distributions Corresponding to Time Series," *Journal of the American Statistical Association*, December, 1931, p. 407.

The means of the two distributions, 96.0 and 104.1, differ significantly. The explanation, aside from the question of a stricter criterion in the case of the second frequency distribution, is probably that the criterion, $9,000 \geqq V' \geqq 11,000$, caused the first distribution to be drawn from one set of days, the second from another set or days not identical with the first. That is, the first samples were from the universe, "W's for active stocks on first-group days," and the second samples were from the universe "W's for active stocks on second-group days." As we desire samples from the universe "all W's without regard to time occurrence," it is perhaps legitimate to combine the two frequency distributions in spite of the different means.

The combined frequency distributions comprise 670 observations with a mean k_1 of 6.73 class intervals.

$$n = 670,$$
$$k_1 = 6.73, \qquad a = 3.60 \text{ class intervals},$$
$$k_2 = 10.90, \qquad\quad = 54.0 \text{ full lots},$$
$$k_3 = 39.35,$$
$$b = 1.02,$$

If the transformation, $x' = x/3.60 - 1.02$, is used, the fit is very poor. $P < .01$. $F(x')$ is zero for $x = 0.097$ class intervals, or 145 shares, which is plausible: the sponsors of a 10,000-shares-a-day stock usually have enough interest in it to arrange at least one transaction per day. But the value of a is beyond reason. 3.60 class intervals = 5,400 shares. This would mean only two *independent* transactions per day. The large a comes about as a result of the flattening and skewness of the frequency curves. This, in turn, may be due to (a) poor graduating properties of the moving average, or (b) combining frequency curves with different means, or (c) sampling error in third moment, affecting k_3.

Another transformation is $x' = x/a - a$. This ensures that $F(0) = 0$. a is found to be 2.715 from the formula,

$$a = \tfrac{1}{2}k_1 \pm \sqrt{\tfrac{1}{4}k_1^2 - k_2}.$$

This is a more reasonable value: 2.715 class intervals = 40.73 full lots. But it is still too large to accept as other than an empirical constant. $k_1(x') = m$ is found from the relationship

$$m = \frac{1}{a} k_1(x) - 1.$$

The fit is fairly good: $P = .15$.

TABLE VI
Curve Fitted to Combined Frequency Distributions

W	x	$x'(1)$	Actual	Calculated	$x'(2)$	Calculated
0	0	−1.026	0	0	−1.0	0
15	1	− .749	2	24.7	− .632	17.9
30	2	− .472	38	49.8	− .264	40.2
45	3	− .195	66	68.8	+ .104	60.8
60	4	+ .082	87	79.1	+ .472	75.4
75	5	+ .359	97	81.0	.840	81.2
90	6	+ .636	78	76.4	1.208	80.5
105	7	+ .913	80	67.0	1.576	73.3
120	8	+1.190	61	56.7	1.944	62.6
135	9	+1.467	41	45.5	2.312	50.6
150	10	+1.744	44	35.2	2.680	39.0
165	11	+2.021	20	26.3	3.048	28.7
180	12	+2.298	15	19.0	3.416	20.5
195	13	+2.575	14	13.4	3.784	14.0
210	14	+2.852	13	9.23	4.152	9.37
225	15	+3.129	5	6.21	4.520	6.05
240	16	+3.406	1	4.09	4.888	3.77
255	17	+3.683	3	2.64	5.256	2.33
270	18	+3.960	1	1.68	5.624	1.39
285	19	+4.237	1	1.05	5.992	.83
300	20	+4.514	0	.64	6.360	.50
315	21	+4.971	2	.39	6.728	.28
330	22	+5.068	0	.23	7.096	.17
345	23	+5.345	1	.14	7.464	.06

$$\left.\begin{array}{l} x'(1) = 0.277x - 1.026 \\[4pt] F(x) = A\dfrac{0.839^{x'}}{\Gamma(x' + 1)} \end{array}\right\} P < .01$$

$$\left.\begin{array}{l} x'(2) = (0.368x - 1) \\[4pt] F(x') = A\dfrac{1.480^{x'}}{\Gamma(x' + 1)} \end{array}\right\} P = .15$$

a is clearly a function of V. It would be very desirable to work out the characteristics of the distribution for other values of V. Then, with $a(V)$ being known, the probability of any given volume v could be calculated.

<p style="text-align:center">*   *   *</p>

What we have to use in practice is $U = v/V$, where V is for the preceding nine days. The analysis presented below was worked out some time ago before the above quasi-Poisson distributions were worked out. The simple logarithmic transformation, $\lambda = \log v - \log V$, yields *approximately* symmetrical frequency distributions, suitable for study although skewed somewhat towards the left. The measure of dispersion used was the arithmetic deviation of λ about 0 because it is easy to compute and because it is not affected by extreme values to the same

extent as σ. The results are shown in Table VII. Groups 1–11 include
3,516 λ's of Consolidated Gas, General Motors, U. S. Steel, American

TABLE VII

ARITHMETIC DEVIATION OF log v FROM log V FOR SIX STOCKS 1929–1931, AND
THE MARKET AS A WHOLE, 1929–1932

Group number	Center of class interval	N	A.D.
1	160*	363	0.219
2	257	519	.202
3	350	582	.184
4	443	567	.182
5	545	376	.177
6	646	243	.173
7	748	203	.168
8	900	322	.162
9	1,090	179	.163
10	1,290	107	.167
11	1,380	55	.195
		3,516	
12	5,000	4	0.117
13	6,000	9	.255
14	7,000	6	.11
15	8,000	23	.153
16	9,000	24	.134
17	10,000	33	.153
18	12,000	50	.123
19	14,000	63	.121
20	16,000	89	.101
21	18,000	103	.117
22	20,000	102	.106
23	22,000	72	.104
24	24,000	56	.112
25	26,000	60	.130
26	28,000	43	.105
27	30,000	39	.118
28	32,000	55	.103
29	34,000	46	.095
30	36,000	56	.065
31	38,000	53	.064
32	40,000	53	.052
33	42,000	71	.068
34	44,000	36	.069
35	47,500	68	.085
36	52,500	30	.122
37	57,500	21	.193
38	62,500	9	.154
		1,274	

* Full lots.
A. D. is observed arithmetic deviation of log v from log V.

Can, General Electric, for 1929–1931. Groups 12–38 include 1,274 observations from the market as a whole.

The results may be questioned because the individual stocks are widely known trading favorites whose volume gyrations may not be typical of the average stock. The period covered is one of the most erratic in the history of the market and volume fluctuations are to that extent not typical. Nevertheless it is hoped that the *relative* decrease of A. D. of λ with increased V is correctly measured

These observations clearly show a tendency for A. D. (λ) = $A(V)$ to decrease with increased V. The observations, plotted on double log paper, lie about a straight line of slope -0.15192. The equation connecting A. D. (λ) and V, is

$$\log_{10} A = 9.9740 - 0.15192 \log V.$$

A table of graduated values is shown.

We are now in a position to measure individual deviations λ in terms of the standard unit of measure, namely, the $A(λ)$ usually associated with V.

Example: $v = 2,500,000$, log $v =$ 6.3979,

$V = 1,500,000$, log $V =$ 6.1761,

λ = $+ 0.2218$,

$A =$ 0.1085,

$ρ =$ 2.04.

Nomographs may be constructed so that the volume index, $ρ$, may be read off directly for given v and V.

TABLE VIII

VALUES OF A AS A FUNCTION OF V: CALCULATED

V	A	V	A	V	A	V	A
1,000	0.3298	30,000	0.1967	250,000	0.1427	2,000,000	0.1040
2,000	.2968	40,000	.1885	300,000	.1419	2,500,000	.1005
3,000	.2791	50,000	.1820	400,000	.1327	3,000,000	.0977
4,000	.2672	60,000	.1771	500,000	.1283	4,000,000	.0935
5,000	.2593	70,000	.1729	600,000	.1248	5,000,000	.0904
7,500	.2428	80,000	.1695	700,000	.1219	6,000,000	.0880
10,000	.2324	90,000	.1665	800,000	.1195	7,000,000	.0859
15,000	.2168	100,000	.1638	900,000	.1174	8,000,000	.0842
20,000	.2092	150,000	.1540	1,000,000	.1154	9,000,000	.0827
25,000	.2022	200,000	.1475	1,500,000	.1085	10,000,000	.0814

Table IX gives the frequency distribution of $ρ$'s for a number of stocks during January–September, 1934. It agrees but poorly with the normal curve of distribution, as shown in the third column. The

A. D.'s of all five stocks are considerably less than the theoretical 1.0, owing to the comparative quietness of the March–April market compared to 1930–1932, the period from which the theoretical distribution was derived. This means, not that the unit of measure no longer applies, but that the character of the market changed, and the process of measurement registered that fact.

TABLE IX

FREQUENCY DISTRIBUTION OF ρ's OF X, GJ, S, GOR AND MTY MARCH 1– APRIL 28, 1934

ρ*	Actual	Theoretical
36		0.4 ⎫
33	1	0.7 ⎬ 2.4
30		1.3 ⎭
27		2.4
24	3	3.9 ⎬ 12.5
21		6.2
18	5	8.5
15	6	12
12	10	15
9	13	19
6	18	21
3	29	23
0	35	24
− 3	20	23
− 6	32	21
− 9	26	19
−12	17	15
−15	13	12
−18	8	8.5
−21	3	6.2
−24	5	3.9
−27	4	2.4
−30	2	1.3 ⎫ 2.0
−33		0.7 ⎭

* In units of 0.1 A. D

Measurement of stock volume in A. D. units permits the calculation of the correlation between the volume of trading in different stocks and the market as a whole. Table X shows the results of correlating ρ's for U. S. Steel, General Motors, Goodyear, Sears, and McIntyre, with the total market. None of the correlations are very high (Table X) and even these are partly spurious because each stock volume is included in the aggregate with which it is correlated.

These correlations are much smaller than those of price changes; volume is much more characteristic of the individual stock, and much less a matter of sympathetic response to the market as a whole, than day-to-day price changes.

TABLE X

CORRELATION OF VOLUME OF TRADING IN FIVE STOCKS WITH THE TOTAL
MARKET, MARCH 1 TO APRIL 28, 1934

Unit: .1 A. D.

	X	GM	GOR	S	MTY	Market
Mean ρ	−2.4	−2.2	−1.9	−3.4	−3.4	−2.6
Theoretical A. D. from Origin	10	10	10	10	10	10
RMSD	8.9	11.8	8.8	10.3	13.6	10.1
σ from mean	8.6	11.6	8.6	9.4	13.2	9.8
A. D. from mean	6.9	9.2	6.9	7.5	10.6	7.8
Correlation with market	.55	.37	.46	.31	.12	

RMSD = root-mean-square-deviation from origin.

III. VOLUME AND PRICE

The discussion so far has treated price and volume separately. It
has proved possible (at least from a statistical standpoint) to regard
stock transactions as distributed collectively at random. A scale for
measuring the technical significance of volumes was devised, on the
theory that equally probable volumes were equally significant from a
technical standpoint. This probability approach apparently makes
progress in the question of measurement and units. Nevertheless, the
analysis, while an advance over commonly employed methods, does
not go far enough. Realistically, the market analyst constantly thinks
of volume in relation to price changes, and vice versa. Was the volume
too small, considering the size of the advance? Did prices fail to show
gain commensurate with the volume? Did stocks "churn" (heavy
volume and confused movement with no net gain)? Were the price
drop and the volume large enough to indicate a "climax"?

The concept of excess or deficiency in volume will be sharpened by
considering volume relative to price.

The following study covers only a short period in the first quarter
of 1934, but a close enough connection is shown between volume and
price to indicate that this is a fruitful method of approach. We may
write an equation between volume and price thus:

$$(1) \quad v = \phi(t) \cdot (ax + b_1c + b_2c^2 + g_1d + d_2g^2 + h) + \text{residuals},$$

where ϕ = a factor measuring the general level of activity, and also
adjusting for slow changes in the relation between v and
the quantity in parentheses. Functional form unknown,
indeterminate.

( ) = a function of prices.

x = the "excess range," that part of the daily range not con-

tributing to the net change; equals the range less the absolute value of the net change. All "gaps" are done away with by extending the range up or down to the close of the preceding day when gaps occur.

c = the net change.

d = the net change for the preceding day.

h = a constant.

v = the daily volume of trading on the New York Stock Exchange, in thousands of shares.

The second degree terms are inserted in order to allow for a curvilinear relationship between change and volume. Large volumes are associated with large price changes, both positive and negative, small volumes with small price changes.

To strike ϕ out of the equation would clearly be unrealistic. Yet its presence increases the statistical difficulties. One way would be to choose a period when the market is relatively quiet, and assume that ϕ remains relatively constant, or changes slowly. Then, except for second order effects, we can get rid of ϕ by using the variate difference method. The method here adopted was to approximate ϕ. It was assumed that $\phi(t)$ was proportional to V'/R', where V' is the centered nine-day moving average of volume, and R' the centered nine-day moving average of the high-low range. V'/R' is the average number of shares per point of range. (1) becomes

$$(2) \qquad \frac{v}{V'} = \frac{F(x, c, d, h)}{R'} + \text{residuals};$$

and thus both sides of the equation are, to a first approximation, corrected to comparable units of measure. For multiple correlation analysis we use

$$(3) \qquad \frac{vR'}{V'} = v' = ax + b_1c + b_2c^2 + g_1d + g_2d^2 + h + \text{random errors}.$$

See Table III.

The following values were found:

$$(4) \qquad \begin{aligned} v\phi^{-1} &\cong F = 5.23x - 0.568c + 0.0221c^2 + 0.224d + 0.0080d^2 + 803, \\ R^2 &= .523, \quad R = .72. \end{aligned}$$

As is well known, this process does not give the surface of best fit.[9] A method of determining the hypersurface of best fit has not, as far

[9] See, for example, Henry Schultz, *Statistical Laws of Supply and Demand with Special Reference to Sugar*. Also Merriman, *The Method of Least Squares*, p. 127.

as the writer knows, been worked out, but a plausible subterfuge suggests itself, which reduces the problem to one of obtaining the line of best fit. Consider $F \equiv z$ as the (single) independent variable, and determine the best fit line,

$$(5) \qquad\qquad F_1 = mz + n,$$

by the usual methods. If we suppose that the law of errors in the x-direction is about the same as that in the y-direction, i.e., that $p = 1$, we get $m = 1.533$. As a matter of fact, we have no basis for selecting any particular p, and we could have chosen a value approximately equal to 1, which would have given the value $m = 1.50$. This value was taken, and the value of n is then -733. Equation (4) then becomes

$$(6) \qquad F_1 = 7.85x - .85c + 0.0331c^2 + .336d + .01193d + 432.$$

The values of F_1 are shown in the last column of Table III. A decimal point has been omitted and they should be read as 1.045, 1.590, etc. v/F_1 is shown in column 6, the unit here being 10,000 shares. Column 6 was next graduated, the result being ϕ_2, the second approximation to ϕ.[10] The ratio of v/F_1 to ϕ_2 is designated ω_1 and may be looked upon as the volume density adjusted for various factors. This should furnish a more usuable measure of excess or deficiency in volume.

It is likely that b_1, b_2, g_1, and g_2, change with the stage of the market. During accumulation we would expect b_1 to be positive, or with a small negative value, because declines tend to occur on small volume (for the given price change) during such periods; b_2 to be rather small; and g_1 to be negative, because declines attract investment buying. During mark-up, b_1 would perhaps have a larger positive value, because trading for the short moves increases with sharp price advances; g_1 perhaps a small positive value. During distribution, b_1 would become negative, as volume then tends to come out on declines, and demand drops off on the rallies; g_2 ought to be decidedly positive, because sharp advances encourage unloading by investors and are the necessary condition for unloading by pools. Characteristic relationships should exist between the volume factor and the concurrent price change, and the price change of the preceding day. Thus a sharp advance, followed by a large ω and a small price advance, or an actual decline, should indicate distribution.

IV. MARKET PROGNOSIS

This paper is intended to deal only with the metrical aspects of price and volume analysis, the barometric aspects being reserved for

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[10] The process can of course be repeated indefinitely, and will usually approach a limit.

later consideration. However, the general problem may at least be formulated, viz., to discover functions $G(p, v)$ of price and volume such that the probability of rising prices is very high for $G > G_2$, and such that the probability of falling prices is very high for $G < G_1$. It is believed that the metrical devices here presented considerably simplify the next stage of the study.

The direct functions of p and v are in all likelihood so complicated as to defy discovery, but if p and v be first transformed into p' (isopoints) and ω or ρ (volume density or volume index), it may be fairly simple to find functions $H(p', \rho, \omega)$ which are barometric. For example, if the one-day rallies of a comprehensive market average out of downtrends are measured in isopoints laid off on the y-axis, and the concurrent volumes measured in ρ-units along the x-axis, it appears that a curve c_1 can be drawn so that, for points lying within it, the probability of further advance is fairly high.

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ANNUAL SURVEY OF STATISTICAL TECHNIQUE: TRENDS AND SEASONAL VARIATIONS

By Paul Lorenz

Introduction.—In the July numbers of Volumes I (1933) and II (1934) of Econometrica, W. A. Shewhart and G. Darmois have already reported upon the progress of statistical technique. Both have stressed the difficulties encountered in such a survey if it attempts to cover too large a field. The point is well taken, and I shall, therefore, confine this survey to a special field within mathematical statistics, namely, trend and seasonal variation as tools in business cycle analysis.

THE MATHEMATICAL METHOD IN BUSINESS CYCLE ANALYSIS: LIMITATION IN ITS RELATION TO OTHER MATHEMATICAL STATISTICS

Mathematical business cycle analysis is a part of mathematical economics, but one cannot say that it has evolved from the older "économie pure." Perhaps it may even be said to have originated from a conscious contrast with the latter, and it is only recently that several authors have tried to bridge the two. If we disregard these recent endeavors and try to establish definitely the conceptual limitation of mathematical business cycle technique, we can differentiate three sharply defined approaches according to the character of the problems set and the manner of attack:

I. the school of classical mathematical political economy,

II. the method of theoretical probability, and

III. the method of business cycle analysis.

I. For the problems of the school of classical mathematical economics, whose representatives are Cournot, Dupuit, Gossen, Jevons, Walras, and others, it is characteristic that they did not evolve from the statistical-numerical observation of economic processes and, furthermore, that the results of the deductions made from the established premises are not proven by the statistical-numerical observation of reality. Classical mathematical economics is pure theory and in character resembles the non-mathematical theoretical type. It contents itself with assuming the quantitative comprehension of the ideas with which it is concerned and making its deductions from the definition of ideas by means of mathematical conclusions. In many of its treatises one does not find a single statistical chart representing values of observation.

II. The method of theoretical probability stands in direct opposition to this. The fields of research, the manner of procedure, and the

aims of the two, are absolutely different. The field of investigation of the school of theoretical probability is statistics. Every single problem is linked either at its beginning or its end to a series of observations. One can scarcely find a textbook or a monograph of this school in which the construction of ideas is not placed in relation to, and proved by, series of statistical observations. The two schools are similar in the predominance of deduction in their manner of thinking. But in the school of theoretical probability it is a question of clarifying certain series of observations, more exactly, a question of understanding them on the basis of a general principle. This general principle is the idea of probability, as it was first evolved from the theory of games of chance and then later extended to the theory of observation of errors in physical measurements. It is, therefore, a question of analogies, of a transference of experiences gained in certain fields which, however, are foreign to the statistics of social mass phenomena and of results reached through reflection in the statistics of these mass phenomena.

If this transference is justified, a rich harvest may be expected from the beginning in the field of statistics of mass phenomena, since the theory of games of chance has from earliest times been most carefully constructed and reconstructed. On the other hand, the theory of observations of errors has also successfully stood many tests on the ground of reality. Moreover, through this investigation in the distribution of statistical quantities, not only is a description and analysis intended but an explanation is given why and how this and no other structure of mass evolved.

As criteria of whether or not these statistical mass observations are subject to the laws of distribution and, if so, to what laws, different methods are available: the dispersion method of W. Lexis and L. von Bortkiewicz, the χ^2 method of K. Pearson, and the ω^2 method of R. von Mises.

In developing his theory, Lexis ("Concerning the Theory of the Stability of Statistical Series" in *Jahrbücher für Nationalökonomie und Statistik*, Jena, 1879) worked out a differentiation of statistical series according to the following criteria:

... in some cases the items of a statistical series, in spite of much irregularity in detail, show on the whole a pervading tendency towards change in a definite direction; these series correspond in a measure to a historical development and and may therefore be called "evolutionary." In other series we find, to be sure, a certain relationship of direction of change in related items but, on the whole, only an oscillation which graphically would be represented by irregular wavelines, for which reason these series may be termed "undulatory." If these movements relative to wave-length and amplitude recurred at regular intervals, we should have "periodic" series. If, on the other hand, the individual values of a series move up and down without connection in a certain space, the designation

'oscillatory' would seem justifiable. A sharp delimitation of this class of undulatory series is just as impossible as an exact separation of the latter from the evolutionary type. But, under this general idea of the oscillatory series, there would come a class which we could place in an entirely separate category in contrast with all the others, namely, the 'typical' series, whose peculiarity consists in the fact that the individual values are inexact representations of a constant basic value which is expressed only in purely accidental deviations.

The typical series is the only one to which the method of theoretical probability may be applied. Thus, research by way of theoretical probability is narrowly limited, for most economic series have a pronounced evolutionary character. Only if it were possible to choose from these series those of an evolutionary trend by dividing them, so to speak, into two constituents, an evolutionary and a constant, and if, furthermore, we could presuppose of this constant part that it is more than the formal result of an arithmetical calculation, namely, an *a priori* existing superstructure on the evolutionary series of typical character, then and then only would the possibility exist of a treatment by theoretical probability of many economic series (Oskar Anderson, "Zur Problematik der empirisch-statistischen Konjunkturforschung" and "Die Korrelationsrechnung in der Konjunkturforschung," Number 1 and Number 4 of the *Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung*, Bonn, 1929.)

III. Mathematical business cycle analysis is differentiated from the school of classical mathematical economics by the presence of and emphasis on the mathematical point of departure. Every problem in this field of research is based upon actual numerical-statistical observation. The methods developed serve the sole purpose of advancing the investigation of numerical series of observations.

Here it is similar to the method of theoretical probability in statistics, but differs from it in its inductive procedure and especially in the fact that it establishes no general principle at the beginning of its investigations, that is, no principle to which it would attempt to subject the phenomena to be examined as the research by theoretical probability considers its objects in view of the governing idea of probability. Therefore, we have here not to do with a fundamental transference of analogies from one field of science to another.

This is a disadvantage for the scope of the method, but otherwise a distinct advantage, since from the very beginning it is not limited to a class of a small compass of series of observations as the method of theoretical probability is. Of course, it must be expected from the very start that, by means of the mathematical methods thus far developed, much may be deduced from one series of observations, only little from another, and perhaps nothing whatever from a third.

Since there exists no general criterion for the limits of applicability of the new methods, one using them must not fail to employ common sense while doing mathematical calculations on series of observations. In addition, a certain feeling for possible economic combinations must be presupposed and cultivated by a sufficient study of economics. An American author (Edwin B. Wilson, "Mathematics and Statistics," in *Journal of the American Statistical Association*, March, 1930) says very pertinently,

For, mathematics is a queer horse and all too easily runs away with its rider; and then there is such a satisfaction in trying its various gaits in all sorts of roads that many a rider has gone off in almost the opposite direction from the path he should have followed in his pursuit of the solution to some scientific problem; he may have ridden over his solution to some purely fantastic goal.

In a general way, one can describe the method somewhat in the following manner: since the size of the individual values of an economic series is generally postulated by causes of a different nature, we shall try to group the causes to be considered according to certain points of view and form an idea as to how each one of these groups of causes *per se* would react upon the economic series considered. For the purpose of illustration, I shall choose an example, namely, the everyday disposition of cars of the Reichsbahn for the transportation of freight as a symptom of the state of trade. This statistical quantity depends upon very different circumstances, which we can classify in four major groups: (1) Gradual transformations in consequence of increase of population, expansion of production and commerce, development of industry, etc.; (2) Seasonal influences; (3) Influences of changing cycles; and (4) Individual influences of an irregular nature (abnormal weather, etc.).

It is impossible to enumerate all these causes and fit them into these four categories, because of their endless variety. A synthetic structure of existing economic series is, thus, impossible. W. M. Persons ("Explanation of the Data and Methods Used in the Index of General Business Conditions," in *Review of Economic Statistics*, Volume I, 1919) therefore reversed the problem and put the question in the following manner:

Is it possible to break down an economic series, given as a whole, into its components in such a way that one is justified in regarding each component as that particular series of economic data which would have been the result if only one category of causes had been at work?

The isolation of changes brought about by seasonal influences in the data of an economic series is not difficult if we may presuppose, as is often possible, that the seasonal influences have repeated themselves

with regularity year after year over a considerable period of time with approximately equal force (we shall come back to this later).

It is much more difficult to isolate satisfactorily the evolutionary changes brought about through gradual transformations in economics. Since the causal connections can only rarely be established, we must generally be content with a formal arithmetical consideration which consists in attempting to determine a basic direction of the movement, a trend.

When we have determined the seasonal variation and the trend of an economic series, we get by elimination a residual series, which we take as the symbol of the cycle of the respective economic phenomena, always presupposing that we have taken into due consideration the reactions of irregular phenomena, such as abnormal climatic conditions, strikes, and so forth.

A decisive factor in the recognition of the method of mathematical business cycle analysis and, thus, of a considerable part of business cycle analysis itself is whether or not we are prepared to recognize figures of economic meaning in the components of series of observations which have been found on the basis of formal mathematical principles.

We must stress the fact that the objects of research in business cycle analysis are always series of observations which extend over longer or shorter periods of time. From this it follows that this field of research deals principally with the investigation of variable processes of economic phenomena. In this respect it stands, therefore, in contrast with the school of mathematical economics, whose problems are problems of economic statistics. But, for this very reason, it differs also from the method of theoretical probability in statistics, since the applicability of this latter has been admitted as certain by some authors but has not yet been proven. The writer believes that he should deny the applicability of the theory of probability to problems of economic development, for he considers the postulates unfounded. The assumption that each individual item of a time series is an "accidental variable" in the real sense of the word, that is, in the sense of the results of games of hazard, does not appear to him at all obvious or even probable. The same thing applies in a greater degree to the idea of "mathematical expectancy" of individual observations in series of economic data arranged according to time, which concept is equally indispensable for the application of the theory of probability.

THE TREND

What is the trend? Technically speaking, the trend is the representation of the evolutionary changes in an economic phenomenon brought

about by the gradual transformation of economic institutions. Expressed methodologically, it is a line which obeys a simple mathematical law that gives only the general tendency of the economic series but not its details. In order to be able to work successfully with the trend one must make two requirements, one formal and the other technical.

The formal requirement is as follows: the trend fitting an economic curve must give the main tendency of the curve as perfectly as possible, but must not introduce any strange elements of movement into its course.

The requirement concerning the matter consists in being able to recognize in the trend the geometric symbol of certain economically significant concepts.

The above-mentioned work of W. M. Persons was a pioneer in modern trend research. Of importance for the introduction of these new ideas in Germany was the monograph of Hermann Hennig, "Die Analyse von Wirtschaftskurven" (Sonderheft 4 der *Vierteljahrshefte zur Konjunkturforschung*," Berlin 1927). The article by the writer, "Denkrichtungen in der mathematischen Oekonomie und Statistik" (*Jahrbücher für Nationalökonomie und Statistik*," 136 Band, Jena, 1932), takes a critical attitude. Of most recent date we may also name: Edwin Fricky, "The Problem of Secular Trend," in *Review of Economic Statistics* (Cambridge, Mass., 1934) and William Hay, "Forecasting of Trends Simplified for Proper Use of Population Statistics" (*Annalist*, New York, 1934). The book of S. Kuznets, *Wesen und Bedeutung des Trends*, (Heft 7 der *Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung*, Bonn, 1930) gives an historical summary of the development of the trend idea reaching far into the past. The textbooks of statistics in which the trend has been given exhaustive presentation need not be specifically enumerated.

The trend concept originates from the fact that almost all series investigated by business cycle analysis extend over a certain period of time. If, for instance, we look at the series of the German harvest figures for rye (in thousand tons) and the graphical representation (see Figure 1) and assume the task of describing the line of this series, the first thing we notice is that the series rises on the whole. It does not go up evenly year by year, and in some cases even falls below the preceding year. But, on the whole, it does rise. It possesses a persistent tendency extending over a rather long period of time, which, of course, is disturbed at times by influences of short duration, but which always overcomes these short-lived disturbances.

It is a question of describing this tendency in a simple manner with the help of mathematics. Is it a straight line or a curve? If it is a curve,

does it follow a mathematical law, and which law? These are questions, which, if put in this manner, are meaningless. Why should the growth of harvest figures follow a mathematical law? That would be to presuppose that, quite aside from disturbances, the size of the harvest would be a function of time, of the number of years. The height of a body thrown into the air is a function of time simply formulated. That has been solved by thousand-fold repetititon of the experiment of throwing in different places and at different times, and the mathematical formula which one could give to the elevation admits, in a simple way, deductions from the past as to the forces active during this process. A direct contrast to the circumstances here described which are

DEUTSCHLANDS ERNTEERTRAGE AN ROGGEN, MILL. t, 1894 ·· 1914.
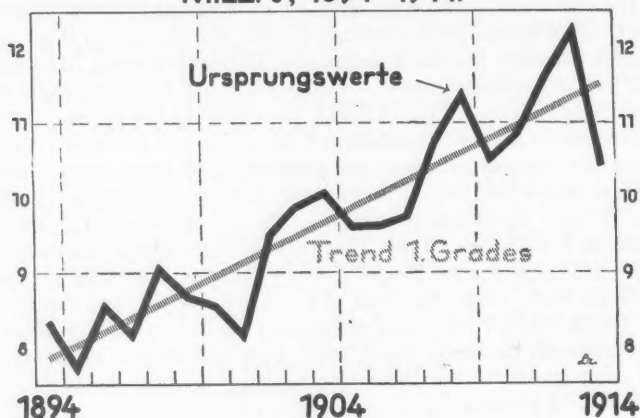
FIGURE 1

present when a body is thrown into the air are the circumstances which must be considered in the figures of the German harvests from 1894 to 1914. The process described in this numerical series takes place only once and can not be repeated. The question whether or not a causal necessity is at the base of it, in consequence of which fact it would have to take a straight trend in its first approach, has here an entirely different meaning from the throwing experiment. It is possible that the forces which have an influence on the size of the harvest—beside the extent and intensity of agriculture and fertilization, and climatic conditions, these consist of the conditions and changes in the increase of population, prices, foreign commerce, politics, and so forth—bring about a tendency toward either a straight or a curved rise for harvest

returns. Whether or not this has been the case in the period of time under consideration is something we can only find out by studying the curve, but never by speculation or by experiment. Therefore, it would be senseless to generalize the result.

If this is the case, the description of the central trend of a statistical curve tells us much less than we would like. From this, unfortunately, the fact results that, even though the past trend of a series may be graphically described, the future one does not necessarily follow the same curve. That means that the line of central trend of a statistical curve is not subject to extrapolation without an exhaustive substantial motivation. In spite of all these defects, it is of fundamental importance and of tremendous interest for business cycle analysis.

THE MATHEMATICAL FORMULATION OF THE TREND

That the mathematical formulation of the trend can not be done without a certain arbitrariness lies within the very nature of the problem. First of all, we must decide upon a method of function. In individual cases, certain salient characteristics of the empirical curve may suggest the choice of a definite functional system. Kuznets, for instance, believes that he can deduce from the trend of many curves of production the following requirements for the curves which are supposed to represent the central trend: (a) the curves must rise and fall continually; (b) when the curve rises, it approaches a definite border value, and its percentage increase must therefore decrease; (c) the absolute increase depends in part on the level already attained and in part on the distance from the border value. Such a law of growth is described by a number of curves. The simplest of these is the logistic curve, with the equation:

$$(1) \qquad\qquad y = \frac{L}{1 + e^{a-bx}},$$

in which L is the limiting value, x the time, and y the dependent variable.

The absolute increase in this curve is proportional to the product of the already attained volume with the distance from the border value. It is, namely,

$$\frac{dy}{dx} = y(L - y) \cdot \frac{b}{L}.$$

We see from this term that dy/dx for a certain curve can never change its sign. This means that the curve represented by (1) rises continuously or drops continuously and, therefore, cannot take an undulatory course. But, since many series, particularly almost all price

series, show a wave motion, and since price series play an important part in business cycle analysis, it follows that we cannot manage with the logistic curve alone in mathematical business cycle analysis.

Of a much more general nature are the integral rational functions whose graphs are, as we know, designated as parabolas. Their structure, besides, is much simpler and clearer. For this reason they are more easily handled from an arithmetical-technical point of view, a circumstance which weighs heavily in their favor, since fundamentally no objection can be made to the integral rational functions. Moreover, they have the advantage that the number of constants, on which the flexibility of the curve depends, can be increased in an easy and simple way. They contain the important individual cases of the arithmetical means, of the straight line, and of the common or throwing parabola, and, finally, their constants can be easily determined with due consideration of all observation values. This is not the case in most of the other functional types, where we must work very often with the "makeshift of selected points," which sometimes is called, without justification, the "method of selected points."

The sum of these advantages, which is not possessed in the same measure by any other kind of function, forms a sufficient reason for using the integral rational functions for the mathematical analysis of economic curves and for improving their usefulness and applicability as much as possible.

The nearest form which we can give to an integral rational function of k degree is

$$(2) \qquad y = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k.$$

Experience, however, teaches us that this manner of representation makes working with this function both laborious and obscure, and that it is far more advantageous to give it the following form:

$$(3) \qquad y = a_0 + a_1 X(x) + a_2 X(x) + \cdots + a_k X_k(x).$$

Here $X_i(x)$ is an integral rational function of the i-th degree of x, and we can subject the $X_i(x)$ also to the following requirement, which makes them into the so-called orthogonal functions.

$$\sum X_i(x) = 0,$$

$$(4) \qquad \sum_x X_i(x) \cdot X_j(x) = 0 \text{ for } i \neq j, \text{ and}$$

$$\frac{1}{N} \sum_x X_i^2(x) = 1,$$

when N is the number of the observed values of the statistical series subjected to analysis. The form of the functions X is completely determined by these equations, if one adds also the stipulation that the highest power of x in $X_i(x)$, therefore x^i, is supposed to have a positive coefficient. Practical work with this form is especially advantageous if it is a question of the analysis of statistical series with equidistant values of observations, which, fortunately, is often the case. If the series of the observed values w_x has the range N, for the determination of the constants of the equation we can use the postulate

$$(5) \quad \sum_x [a_0 + a_1X_1(x) + a_2X_2(x) + \cdots + a_kX_k(x) - w_x]^2 \text{ Minimum!}$$

that is, the method of the least square sum. Through partial differentiation according to the coefficients $a_0, a_1, a_2, \ldots a_k$, there results from this

$$a_0 = \frac{1}{N} \sum_x w_x,$$

$$a_1 = \frac{1}{N} \sum_x w_x X_1(x),$$

$$(6) \qquad a_2 = \frac{1}{N} \sum_x w_x X_2(x),$$

$$\cdots \cdots \cdots \cdots \cdots$$

$$a_k = \frac{1}{N} \sum_x w_x X_k(x).$$

The subsitution of these terms for $a_0, a_1, a_2, \ldots a_k$, in (5) makes this expression, as we can easily see, a minimum.

From (6) it follows without further difficulty that it is a matter of indifference in how many items the function of approximation (3) has been put down. We always get the same values for the coefficients $a_0, a_1, a_1, \cdots$.

The function of approximation of the fourth degree is derived from the third degree,

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X_3,$$

simply by adding the item a_4X_4, and would be

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4,$$

with unchanged $a_0, a_1, a_2,$ and a_3. Inversely, the function of approximation of the third degree simply is derived from the fourth degree by leaving off the item a_4X_4.

In the following pages, I give a bibliography[1] of the works that treat of the applicability of the orthogonal integral rational functions for the analysis of statistical series. In going back as far as the fundamental work of Tschebyscheff, it is in the endeavor to make it as complete as possible from a mathematical point of view. But we must keep in mind that neither Tschebyscheff nor the authors named directly after him in this bibliography knew anything whatsoever of the problems of mathematical business cycle analysis. For them the orthogonal functions were solely a tool for the graduation of series and for interpolation, that is, for the tasks with which trend research, as we know it, does not deal. (Compare: Lorenz, "Das Trendproblem in der Konjunkturforschung," in *Blätter für Versicherungs-Mathematik*, (Berlin, 1932) and *ibid.* "Ueber Näherungsparabeln hohen Grades und ihre Aufgabe in der Konjunkturforschung," in *Metron*, Vol. x, Nr. 4, Roma, 1933).

Orthogonal Functions and Their Use in Trend Determination

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2. Tschebyscheff, "Sur les fractions continus." *Œuvres*, ɪ, 1855, 203–230.
3. Tschebyscheff, "Sur une nouvelle série," *Œuvres*, ɪ, 1858, 381–384.
4. Tschebyscheff, "Sur l'interpolation par la méthode des moindres carrés," *Œuvres*, ɪ, 1859, 473–498.
5. Tschebyscheff, "Sur l'interpolation," *Œuvres*, ɪ, p. 541–560.
6. Tschebyscheff, "Sur l'interpolation des valeurs équidistantes," *Œuvres*, ɪɪ, 1875, 219–242.
7. Gram, J. P., *Om Raekkendriklinger bestemte ved Hjaelp af de mindste Kvadraters Methode*, Copenhagen, 1879.
8. Poincaré, H., *Calcul des Probabilités*, 1. Auflage (1896), S. 251., 2. Auflage (1912), S. 280.
9. Pareto, V., "Tables pour faciliter l'application de la méthode des moindres carrés," Communication présentée à l'Assemblée annuelle des Statisticiens officiels et de la Societé Suisse de Statistique, tenue à Lausanne en 1898.
10. Benini, Rodolfo, *Principii di statistica metodologica*, Torino, 1906.
11. Quiqet, A., "Sur une méthode d'interpolation exposée par Henri Poincaré et sur une application possible aux fonctions de survie," in *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge Press, Vol. 2, 1913, p. 385–388.
12. Gram, J. P., "Uber partielle Ausgleichung mittelst Orthogonalfunktionen," in *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker*, 10. Heft, Bern 1915.
13. Esscher, Fredrik, *Ueber die Sterblichkeit in Schweden 1886–1914*. Meddelanden fram Lunds Astronomiska Observatorium, Ser. ɪɪ, Nr. 23, Lund, 1920.
14. Fisher R. A., "Studies in Group Variation," in *Journal of Agricultural Science*, xɪ (1921), Part ɪɪ.
15. Jordan, Karl, "Sur une série de polynomes, dont chaque somme partielle représente la meilleure approximation d'un degré donné suivant la méthode des

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[1] In the compiling of this bibliography, Dr. Carl Boehm, Chief Mathematician of the Institute for Business Cycle Analysis in Berlin and Dipl.-Ing Wilhem Morgner were of great assistance.

moindres carrés," in *Proceedings of the London Mathematical Society*, Second Series, xx (1922), 297–325.

16. Pearson, K., "On a General Method of Determining the Successive Terms in a Skew Regression Line," *Biometrika*, xiii (1923), 296–300.

17. Khotimsky, V., *Smoothing of Statistical Series by Least Squares* (Tschebyscheff's method; russisch), Soviet Press, Moscow, 1925.

18. Isserlis, L., "Note on Chebyscheff's Interpolation Formula," in *Biometrika* xix (1927), 87–93.

19. Lorenz, Paul, "Der Trend; ein Beitrag zur Methode seiner Berechnung und seiner Auswertung für die Untersuchung von Wirtschaftskurven und sonstigen Zeitreihen," *Vierteljahrschefte zur Konjunkturforschung*, 1. Auflage, Sonderheft 9 (1928), 2. Auflage Sonderheft 21 (1931). With tables of the functions $X_i(x)$ from the first to the sixth degree for series up to 80 items, and numerous auxiliary tables.

20. Esscher, Fredrik, "On graduation according to the method of least squares by means of certain polynomials," in *Forsäkringsaktiebolaget Skandias*, Festskrift 1930, Stockholm.

21 Jordan, Karl, "Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate," in *Mitteilungen der ungarischen Landeskommission für Wirtschaftsstatistik und Konjunkturforschung*, Studien Nr. 1. Budapest, 1930 (with tables and auxiliary tables).

22. Sipos, Alexander, "Praktische Anwendung der Trendberechnungsmethode von Jordan," in *Mitteilungen der ungarischen Landeskommission für Wirtschaftsstatistik und Konjunkturforschung*, Studien Nr. 2, Budapest, 1930 (with tables and auxiliary tables).

23. Allan, F. E., "The General Form of the Orthogonal Polynomials for Simple Series, with Proofs of Their Simple Properties," in *Proceedings of the Royal Society*, Edinburgh, l (1930), Part iii, p. 310–320.

24. Lorenz, Paul, "Hohe Korrelationskoeffizienten," in *Allgemeines Statistisches Archiv*, 21. Band, Jena, 1931.

25. Lorenz, Paul, "Ueber gewisse Funktionen, die in der mathematischen Konjunkturforschung eine große Rolle spielen," in *Journal für die reine und angewandte Mathematik*, Band 168, Heft 3, Berlin, 1932.

26. Aitken, A. C. "On the Graduation of Data by the Orthogonal Polynomials of Least Squares," in *Proceedings of the Royal Society*, Edinburgh, liii (1933), with figures.

27. Schenker, O., "Direkter Weg zur Ermittlung von Relationen zur Trendbestimmung," in *Zeitschrift für schweizerische Statistik und Volkswirtschaft*, Bern, 1933.

28. Lidstone, G. J., "Notes on Orthogonal Polynomials and their Application to Least Squares Methods of (1) Fitting Polynomial Curves to Data, (2) Graduation by Weighted Means," in *Journal of the Institute of Actuaries*, Vol. LXIV.

29. Davis, H. T., *Tables of the Higher Mathematical Functions*, Bloomington, Indiana, 1933.

The insight into economic and demographic connections that may be gained by trend research, especially with the help of orthogonal functions, may be illustrated by two graphs (Figures 2 and 3). The first is taken from Wagemann's *Struktur und Rhythmus der Weltwirtschaft* (Berlin, 1931), the second from my above-mentioned monograph, "Das Trendproblem in der Konjunkturforschung."

THE PROBLEM OF SEASONAL VARIATIONS

The seasonal rhythm of economic series, so-called seasonal varia-
tion, is worked out by elementary arithmetical processes, namely,
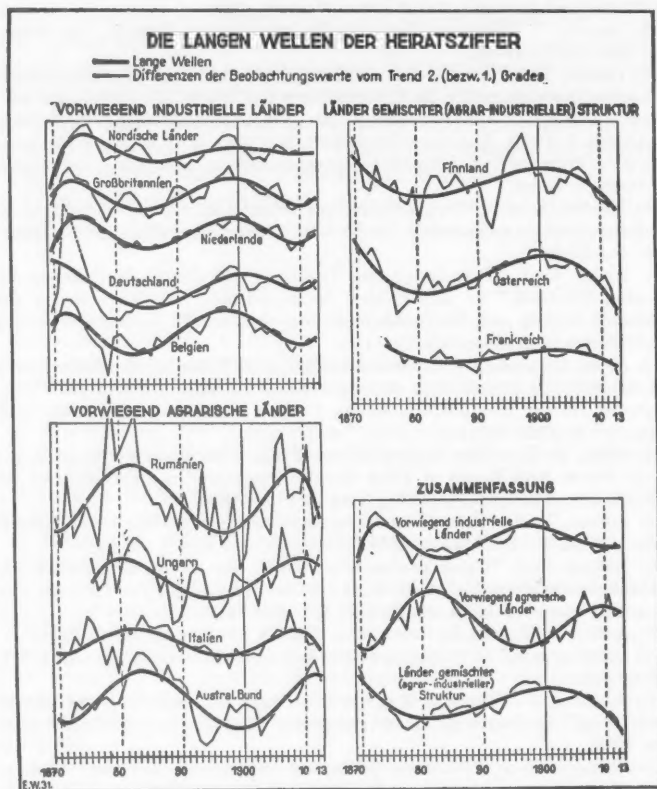through the process of periodogram analysis or link relatives. Where

FIGURE 2—Long cycles in marriage statistics.

it is a question of a very thorough analysis, one can use the Fourier
series, since we deal fundamentally with phenomena having constant
periods.

Concerning the literature which has been published up to the year
1928 on the subject of the method of the mathematical determination
of seasonal variation, we find an exhaustive account in the book of
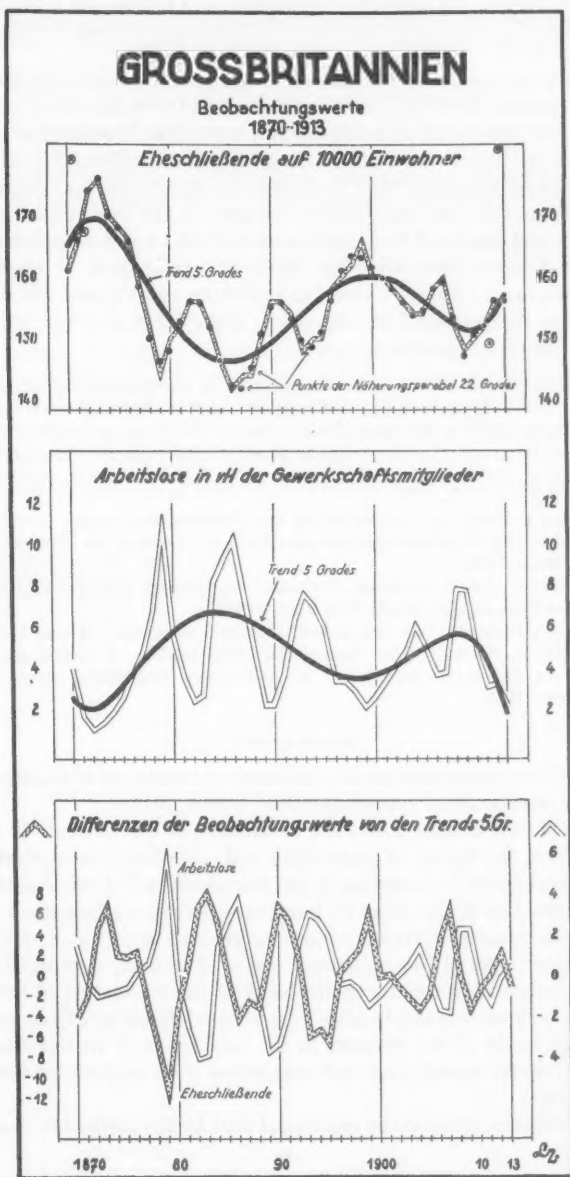
FIGURE 3—Great Britain.

Otto Donner, "Die Saisonschwankungen als Problem der Konjunktur-forschung," in *Vierteljahrshefte zur Konjunkturforschung*, Sonderheft 6, Berlin, 1928.

Much has been written concerning the method of calculation of seasonal indexes. Descriptions of the various methods have been included in all the textbooks of mathematical statistics. Monographs on the subject are found in the most varied magazines, collections, and publications. In Germany the "Verfahren der Relativglieder" introduced by Persons has been most widely accepted. It has been presented, discussed, and criticized, in a vast number of books and occasional pamphlets. Lorenz, especially, has given an explanation of his logical premises in an article, "Ueber das Verfahren von Persons zur Berechung eines Saisonindex," in *Allgemeines Statistisches Archiv*, Jena, 1932.

Of other monographs we mention:

1. Lorenz, Paul, "Die Bestimmungsgrunde für die Saisonschwankungen des Berliner Marktdiskonts in der Vorkriegszeit" in *Vierteljahrshefte zur Konjunkturforschung* (1929) 3. Jahrgang, Heft 4, Berlin—With use of Fourier series.

2. Dupriez, Léon H., "La méthode de calcul des variations saisonnières et des courbes cycliques," in *Bulletin de l'Institut des sciences économiques*, Lowen, 1930.

3. Zrzavy, Franz J., "Ausschaltung von Saisonschwankungen mittels Lag-Correlation," in *Monatsberichte des österreichischen Instituts für Konjunkturforschung*, Wien, 1933.

4. Kuznets, Simon, *Seasonal Variations in Industry and Trade*, National Bureau of Economic Research, New York, 1933.

5. Bachi, Riccardo, "Le regioni nell' economia nazionale," Roma, 1933.

6. Gräbner, Georg, "Der 'bewegliche' Saisonindex. Anmerkungen zum Begriff der Saisonschwankungen," in *Allgemeines Statistisches Archiv*," 24. Band, Jena, 1934.

SUPPLEMENT

The above comments on the mathematical methods of business cycle analysis should be supplemented at several points.

First should be mentioned the calculation of correlations. It originated from the theory of probability and, therefore, its applicability to "typical series"—according to the classification of Lexis—necessarily follows. But it also plays an important part in mathematical business cycle analysis, although here, in general, it is not a question of "typical" but of "evolutionary" series. However, a generally accepted logical basis for the applicability of the calculation of correlations to such cases is still lacking. But we must desist here from a more thorough study of the progress in the calculation of correlations, as well as from an enumeration and recognition of its very comprehensive literature.

Furthermore, it should be mentioned that by the method of numeri-

cal integration and differentiation interesting and economically explicable connections between economic curves of markedly different character can often be found.

As examples, we would cite the following publications:

1. K. G. Karsten, "The Harvard Business Indexes—A New Interpretation," in *Journal of the American Statistical Association*, 1926.

2. Donner and Hanau, "Untersuchung zur Frage der Marktzusammenhänge," in *Vierteljahrshefte zur Konjunkturforschung*, Jahrgang 1928, Heft 2, Teil A.

3. Paul Lorenz, "Eine Differentialgleichung der Wirtschaftsforschung und ihr Integral," in *Blätter für Versicherungsmathematik und verwandte Gebiete*, Berlin, 1929.

As part of the mathematical methods of business cycle analysis in a wider sense, we might also include the problems of cost analysis and of index construction. The very words tell the expert reader that here it is a question of such wide fields of mathematical economic research that they can not possibly be treated as secondary supplements to other related fields of mathematical economic research of a different nature, but require an independent and thorough discussion of their own.

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NOTE SUR LES SYSTÈMES MACRODYNAMIQUES

Par B. A. Chait

Le compte rendu de la réunion de la Société d'Économétrie tenu à Leyde en septembre-octobre 1933, paru dans l'Econometrica d'avril dernier et dont je viens d'avoir connaissance, me porte à en préciser le détail touchant mon intervention aux débats.

En marge du remarquable exposé de M. Tinbergen sur la question: "Est-ce que la théorie des oscillations-harmoniques peut être utile à l'étude des cycles économiques?" j'avais formulé quelques brèves réflexions, à savoir que:

1. Le bénéfice plutôt que le prix est le facteur décisif dans l'activité des entrepreneurs, et pour préciser: le bénéfice unitaire ou la marge bénéficiaire ou, si l'on veut, le rendement financier, bien plutôt que le bénéfice absolu.

2. Les investigations statistiques révèlent des fluctuations à double mouvement d'où se détachent le cycle à période presque décennale et l'onde séculaire dont la période est de quelques dizaines d'années. L'analyse du double mouvement peut s'opérer à l'aide d'un ensemble de conditions conduisant à une équation différentielle du 4e ordre au moins.

3. On ne saurait affirmer *a priori* que les fluctuations économiques sont nécessairement des harmoniques. Des mouvements, autres que les harmoniques, expliquent bien mieux l'irrégularité observée couramment dans les fluctuations.

A propos du travail intéressant de M. Kalecki, "Essai d'une théorie des mouvements cycliques construite à l'aide des mathématiques supérieures."

1. J'avais noté que l'élimination, dès les prémisses, du mouvement de longue tendance restreint singulièrement la portée de son analyse qui s'interdit ainsi d'étudier une des causes importantes de la cyclicité.

2. J'étais frappé de l'influence quasi nulle de la durée d'usure des capitaux, à laquelle les calculs de M. Kalecki paraissent conclure. Cette influence est au contraire très marquée. Le cours accéléré de l'évolution technique d'à présent tend à faire substituer la notion de désuétude économique à celle de dépréciation technique dans la mise au rebut des outillages. La durée de service des capitaux en est abrégée, et cela expliquerait en partie la contraction des périodes cycliques constatée les dernières décades.

3. Le souci de la recherche des causes déterminant les mouvements cycliques semble avoir peu préoccupé cet auteur, alors que justement la raison d'être de ce genre d'études est sans doute d'orienter les efforts vers le plan d'action. Cette tâche est inconcevable sans une exploration préalable, sous ses espèces quantitatives, du problème de la causalité.

Ces observations, données à titre d'indication, trouveront le développement voulu dans l'ensemble d'un travail à paraître prochainement.

<div style="text-align: right">B. A. Chait</div>

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REPORT OF THE MEETING OF THE ECONOMETRIC SOCIETY, COLORADO SPRINGS, JUNE 22–24, 1935

THE Econometric Society held its 1935 American summer meeting at Colorado College, Colorado Springs, Colorado, from Saturday, June 22, to Monday, June 24. Papers were presented by Harold Hotelling, Harold T. Davis, F. Creedy, W. F. C. Nelson, Alfred Cowles III, Wassily Leontief, Nicolas Georgescu, and Charles F. Roos. Attendance at sessions ranged between 30 and 50.

At the opening session on Saturday afternoon, June 22, Professor W. Lewis Abbott, Chairman of the Executive Committee of the School of Social Sciences of Colorado College, presided. Professor Harold Hotelling of Columbia University presented the first paper under the title, "A Basic Defect of Capitalism," saying, in part: "The distribution of wealth is a problem distinct from that of social efficiency. It will, perhaps, never be possible to obtain universal agreement regarding the former, but the latter is capable of treatment by objective methods which should convince all competent persons, regardless of personal economic position." The speaker then gave a new mathematical demonstration that a net social loss, in the sense of a diminution in the total of goods and services, resulted from the substitution of sales and excise taxes for income taxes and, ignoring effects of income distribution on economic well-being, demonstrated that, for any fixed amount of tax revenue derived from a person, this person is left in a better economic condition if the amount of the levy is independent of the manner in which he spends his remaining income. The mathematical demonstration evaded the objections sometimes made to similar arguments on the ground that utility is non-measurable by employing only the concept of indifference loci representing preferences. Professor Hotelling concluded that taxes on incomes, inheritances, and the site value of land, are the only important imposts which do not reduce the efficiency of the economic organization in supplying goods and services. In closing, he argued that governments should attempt to bring about sales of goods at marginal cost, with prices determined in such a way as to contribute nothing whatever to fixed costs, which he said could be better met by taxes of the kind above mentioned.

Nicolas Georgescu of Bucharest, Rumania, presented the second paper, "A Criticism of the Marginal Productivity Theory of Pareto." Dr. Georgescu's paper, which considered difficulties surrounding integrability conditions and offered the suggestion that instead of an indifference direction there may be only an indifference angle (necessarily leading to indeterminateness), was published in the *Quarterly Journal of Economics*, August, 1935.

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In commenting on Professor Hotelling's paper, Alfred Cowles III declared that present income, inheritance, and estate tax rates in the United States made it virtually impossible to maintain intact estates of over $4,000,000, even where none of the income is spent. He said that if income taxes were to be used extensively by governments they would have to be imposed on small as well as large incomes, otherwise the sources of taxation would soon be exhausted. Professor Roos expressed the opinion that Professor Hotelling had made a very important omission in neglecting effects of income distribution on social efficiency, and that this partially invalidated the usefulness of the theory developed. He also enumerated other important neglected factors, such as (1) number of tax payments per year per individual, (2) uses to which taxes are put, and (3) costs of collection of taxes.

On Saturday evening, members of the Econometric Society were entertained in the home of Mr. and Mrs. Alfred Cowles III. Following a buffet supper, there were interesting informal discussions of various economic and sociological questions. On Sunday, members enjoyed a motor trip to Cripple Creek. After lunch at Cripple Creek, they visited the famous Elkton Gold Mine. After an examination of ore formations and a demonstration of methods of removing gold ore, the party descended 1800 feet underground to view a drainage canal which makes operation in the field possible.

At the session of Monday morning, June 24, Charles H. Sisam, Professor of Mathematics at Colorado College, occupied the chair. Professor Harold T. Davis of Indiana University presented the first paper entitled "Some Periodic Aspects of the Stock Market." The objectives of the paper were (1) to exhibit the degree of validity to be attached to the conclusions of harmonic analysis when applied to the Dow Jones Industrial Averages of common stock prices from 1897 to 1914, and (2) to suggest some *a priori* bases for the existence of these harmonics. When a significance test devised by R. A. Fisher for application to harmonic analysis was employed, the periods of 22, 43, and 62 months appeared to have validity, with the 43 months' period dominating. These periods were also found in a harmonic analysis of the Cowles Commission Index of Investment Experience in Common Stocks, covering the years 1872–1900. The speaker suggested that the existence of the periods might be explained on the basis that stock price indexes measure a more or less permanent quadratic form identifiable with industrial energy. The coefficients of this form for the years 1897 to 1914 were exhibited and were shown to be consistent with the energy postulate. In contrasting methods of harmonic analysis, Professor Davis showed that the method of lag-correlations avoided some of the major difficulties of the Schuster method when applied to two series of 300 items with known periodicities.

In discussing Professor Davis's paper, Professor Hotelling declared: "So much is known about the sampling theory connected with regression equations that it is of great advantage to throw the discussion into terms of them whenever possible. The use of auto-correlations, of the 'correlation periodogram,' and of the variate difference method, may all be clarified by reference to regression equations of observations on the next few preceding ones in the same series." He held that in economic data independence is lacking, a condition of R. A. Fisher which must be met when applying his test of significance for periodograms. He urged that, consequently, too much faith should not be put in the continuance of cycles, even when they emerge from periodogram analysis with apparent significance.

Professor F. Creedy of the University of British Columbia presented the second paper, "An Economic Interpretation of the Principle of Least Action and Other Dynamical Theorems." This was an extension of Professor Creedy's essay published in ECONOMETRICA, October, 1934, in which the notion of economic force (defined as proportional to the rate of acceleration of economic activity) was developed and equations similar to those of ordinary dynamics were shown to be applicable to economic problems. Professor Creedy presented a system of economic dimensions and then gave a definition of economic work and economic action, the former being defined as the rate of exchange of goods for money, and the latter as the number of times a unit of goods is exchanged for a unit of money. He then declared that the Principle of Least Action simply stated: "Any economic transaction will take place so that the number of crossings (roughly, exchanges between money and goods) will be a minimum. No one buys except when he must." He closed by giving economic interpretations of some particular branches of dynamics.

In discussion of Professor Creedy's paper, Dr. Georgescu pointed out that it is always possible to find physical analogies for economic phenomena and contended that Professor Creedy should attempt, therefore, to fortify his theory with statistical examples.

At 1:00 P.M. the Society held a luncheon at Bemis Hall, Colorado College, with Thurston J. Davies, President of Colorado College, presiding. President Davies extended the cordial welcome of the college to the Econometric Society. In an after-luncheon address entitled "Painting and Prices," prepared in collaboration with the eminent artist Boardman Robinson, W. F. C. Nelson of the Cowles Commission gave some preliminary results of an examination of J. M. Keynes' suggestion (*Treatise on Money*, II, 154) that "by far the greatest proportion of the world's greatest artists have flourished in the atmosphere of buoyancy, exhilaration, and freedom from economic cares felt by the governing class, which is engendered by profit inflations."

From an analysis of the rate of change of wheat prices from 1260 to 1900 and an index of the 100 greatest painters weighted according to their importance over the same period, Mr. Nelson suggested that the chances of a great painter appearing during a period of rising wheat prices appeared to be almost twice as great as during periods of stable or declining prices.

The Monday afternoon session was presided over by Dr. Charles H. Boissevain, Director of the Laboratory of the Colorado Foundation for Research in Tuberculosis. The first paper was given by Alfred Cowles III of the Cowles Commission, who spoke on "Practical Problems in the Methodology of Multiple Correlation Analysis." Mr. Cowles suggested that much of the difficulty arising in multiple correlation studies of time series is due to high intercorrelations of primary variates and intimated that these difficulties might be avoided by (1) transformation of variates, such as taking the ratio of two highly intercorrelated series or combining them in an index, and/or (2) discarding all but one variable of each group of highly intercorrelated ones through use of λ-determinants or through recourse to a special iterative method for disclosing intercorrelations, which he presented. Mr. Cowles pointed out that, once hypotheses of independence of observations and series were satisfied, tests of significance given in Karl Pearson's *Tables of the Incomplete Beta-Function* could be applied.

The second paper was given by Professor Wassily Leontief of Harvard University, who discussed index numbers from the point of view of indifference curves.

The final paper of the meeting was presented by Professor Charles F. Roos of the Cowles Commission and Colorado College. It gave a résumé of certain of the cost experiments of the National Recovery Administration and will appear in Dr. Roos' book, *The NRA: Some Adventures in Economic Planning*, which is now in the course of preparation.

<div style="text-align: right">H. T. Davis</div>

Cowles Commission for Research in Economics,
Colorado Springs, Colorado

ELECTION OF FELLOWS, 1935

By authority of the Council of the Econometric Society, the Secretary is pleased to announce that, in accordance with the Constitution, the Fellows of the Society have recently elected the four new Fellows whose names and partial bibliographies follow:

R. G. D. ALLEN, University of London, London, England.

"The Foundation of a Mathematical Theory of Exchange," *Economica*, May, 1932.

"On the Marginal Utility of Money and Its Application," *Economica*, May, 1933.

"The Return of Indifference Curves," *Review of Economic Studies*, February, 1934.

"Nachfragefunction für Guter mit korreliertem Nutzen," *Zeitschrift für Nationalökonomie*, March, 1934.

"A Comparison Between Different Definitions of Complementary and Competitive Goods," ECONOMETRICA, April, 1934.

With J. R. Hicks: "A Reconsideration of the Theory of Value," *Economica*, February and May, 1934.

COSTANTINO BRESCIANI-TURRONI, Cairo, Egypt

Studi Sulla Distribuzione Dei Redditi. Si segnalano qui i principali: Sull'interpretazione e comparazione di e riazioni di redditi e di patrimoni, Torino, 1907.

Di un indice misuratore della disuguaglianza nella distribuzione della ricchezza, Palermo, 1910.

"Relation entre la récolte et les prix du coton égyptien," *L'Egypte Contemporaine*, 1930, pp. 57.

"Über die Elastizität des Verbrauchs ägyptischer Baumwolle," *Weltwirtschaftliches Archiv.*, 1931, pp. 40; e infine, in questa stessa rivista, Ottobre 1933, pp. 29.

Inductive Verification of the Theory of International Payments, Giza, 1933.

L'Influence de la spéculation sur les fluctuations des prix du coton, Cairo, 1933.

MORDECAI EZEKIEL, United States Department of Agriculture, Washington D. C.

"Input as Related to Output in Farm Organization and Cost-of-Production Studies," by H. R. Tolley, J. D. Black, and M. J. B. Ezekiel, 1924 (U. S. Dept. of Agriculture, *Department Bulletin No. 1277*).

"Factors Affecting Farmers' Earnings in Southeastern Pennsylvania," 1926 (U. S. Department of Agriculture, *Department Bulletin No. 1400*).

"Factors Affecting the Price of Hogs," by G. C. Haas and M. Ezekiel 1926 (U. S. Dept. of Agriculture, *Department Bulletin No. 1440*).

"Practices Responsible for Variations in Physical Requirements and Economic Costs of Milk Production on Wisconsin Dairy Farms," *Wisconsin Agricultural Experiment Station Research Bulletin 79*, with P. E. McNall and F. B. Morrison.

"Preisvoraussage bei landwirtschaftlichen Erzeugnissen," *Frankfurter Gesellschaft für Konjunkturforschung*. Veröffentlichungen hrsg. von E. Altschul. hft. 9, 32 p. Bonn, K. Schroeder, 1930.

Economic Bases for the Agricultural Adjustment Act, with Louis H. Bean, U. S. Government Printing Office, 1933.

"Studies of the Effectiveness of Individual Farm Enterprises," *Jour. Farm Econ.*, Jan. 1926.

"A Statistical Examination of Factors Related to Lamb Prices," reprinted from the *Journal of Political Economy*, April, 1927.

"Statistical Analyses and the 'Laws' of Price," *Quart. Jour. Econ.*, February, 1928.

"A Statistical Examination of the Problem of Handling Annual Surpluses of Non-Perishable Farm Products," *Jour. Farm Econ.* April, 1929.

"European Competition in Agricultural Production, with special reference to Russia," *Jour. Farm Econ.*, April, 1932.

"Agriculture: Illustrating Limitations of Free Enterprise as a Remedy for Present Unemployment," *Jour. Amer. Stat. Assoc.*, March, 1933 Suppl.

A National Barter System, Oct. 1933.

J. MARSCHAK, All Souls College, Oxford, England

"Die Verkehrsgleichung," *Archiv für Sozialwissenschaft*, Vol. 52, 1924

"Substanzverluste," *Archiv für Sozialwissenschaft*, Vol. 67, 1932.

"Groessenordnungen des deutschen Geldsystems" (with Dr. Walter Lederer), *Archiv für Sozialwissenschaft*, Vol. 67, 1932.

"Volksvermoegen und Kassenbedarf," *Archiv für Sozialwissenschaft*, Vol. 68, 1933.

"Econometric Parameters in a Stationary Society with Monetary Circulation," ECONOMETRICA, Vol. II, 1934.

"Consumption (Measurement)," *Encyclopedia of Social Sciences*, 1929.

Die Lohndiskussion, Tuebingen, 1930.

"Vom Groessensystem der Geldwirtschaft," *Archiv für Sozialwissenschaft*, Vol. 69, 1933.

In voting on nominees, Fellows took into consideration the suggestion of the Council that each Fellow should fulfill the following requirements:

1. He should be an economist.
2. He should be a statistician.
3. He should have some knowledge of higher mathematics.
4. As specified in the Constitution, he should have made some original contributions "to economic theory or to such statistical, mathematical, or accounting analyses as have a definite bearing on problems in economic theory."

Fellows were supplied with ballots containing a bibliography of each nominee and were asked to indicate whether they had read representative works. A surprising result of the vote was the discovery that works of several well-known nominees had been read by only a few Fellows. Indeed, the ballots show that some of the nominees failed of election primarily because their work was to a large extent unknown to the Fellows. This indicates how difficult it was to keep abreast of new developments in econometric research throughout the world when publication was as widely scattered as it was before the founding of ECONOMETRICA. In some instances the writings of a nominee were widely read by Fellows in his own country but were unknown to Fellows in other parts of the world. It appears, therefore, that authors of works which are not published in ECONOMETRICA should endeavor to send out large numbers of reprints. In this way, members and Fellows of the Society can be kept constantly apprised of new discoveries in the rapidly growing field of econometrics.

On the next page is given a complete list of Fellows of the Econometric Society, including the four recently elected.

LIST OF FELLOWS OF THE ECONOMETRIC SOCIETY
SEPTEMBER, 1935

Dr. R. G. D. ALLEN, London, England
Professor LUIGI AMOROSO, Rome, Italy
Professor OSKAR N. ANDERSON, Varna, Bulgaria
Dr. ALBERT AUPETIT, Paris, France
Professor P. BONINSEGNI, Lausanne, Switzerland
Professor A. L. BOWLEY, London, England
Professor C. BRESCIANI-TURRONI, Cairo, Egypt
Professor CLÉMENT COLSON, Paris, France
Professor GUSTAVO DEL VECCHIO, Bologna, Italy
Professor FRANÇOIS DIVISIA, Paris, France
Professor GRIFFITH C. EVANS, Berkeley, Cal., U.S.A.
Dr. MORDECAI EZEKIEL, Washington, D. C., U.S.A.
Professor IRVING FISHER, New Haven, Conn, U.S.A.
Professor RAGNAR FRISCH, Oslo, Norway
Professor CORRADO GINI, Rome, Italy
Dr. GOTTFRIED HABERLER, Vienna, Austria
Professor HAROLD HOTELLING, New York, New York, U.S.A.
Professor JOHN Maynard KEYNES, Cambridge, England
Dr. N. D. KONDRATIEFF, Russia
Professor J. MARSCHAK, Oxford, England
Professor WESLEY C. MITCHELL, New York, New York, U.S.A.
Professor H. L. MOORE, Cornwall, New York, U.S.A.
Professor UMBERTO RICCI, Giza, Egypt
Professor CHARLES F. ROOS, Colorado Springs, Colo., U.S.A.
M. JACQUES RUEFF, Paris, France
Dr. ERICH SCHNEIDER, Dortmund, Germany
Professor HENRY SCHULTZ, Chicago, Illinois, U.S.A.
Professor JOSEPH A. SCHUMPETER, Cambridge, Mass., U.S.A.
Professor J. TINBERGEN, Scheveningen, Holland
Professor FELICE VINCI, Bologna, Italy
Professor EDWIN B. WILSON, Boston, Mass., U.S.A.
Dr. WL. ZAWADZKI, Warsaw, Poland
Professor F. ZEUTHEN, Copenhagen, Denmark

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